Exploring jet properties in magnetohydrodynamics with gravity

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van doctor
aan de Universiteit van Amsterdam
op gezag van de Rector Magnificus
prof. dr. D. C. van den Boom
ten overstaan van een door het college voor promoties
ingestelde commissie,
in het openbaar te verdedigen in de Agnietenkapel
op woensdag 3 april 2013, te 12:00 uur

door

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geboren te Gorredijk
The research reported in this thesis was carried out at the Astronomical Institute ‘Anton Pannekoek’, University of Amsterdam, The Netherlands.

Cover image: A mock diagnostic plot used to find solutions that cross all critical points, shaped in the form of a disc and two bipolar jets. See section 2.3.3 for an example of an actual diagnostic plot used.
# Contents

1 Introduction

1.1 Active galactic nuclei .............................................. 1
  1.1.1 Effect of AGN on galaxy evolution ...................... 3
  1.1.2 AGN classes .................................................... 3
  1.1.3 Unification through orientation ............................. 4
  1.1.4 Intrinsic differences ......................................... 5

1.2 Black hole X-ray binaries ........................................... 5
  1.2.1 Accretion states ............................................... 5

1.3 Mapping of AGN classes onto BHXRB accretion states .......... 7

1.4 Accretion discs ..................................................... 10

1.5 Jets ........................................................................ 13
  1.5.1 Observations of jets ............................................ 13
  1.5.2 Theoretical models of jets ..................................... 16

1.6 Spectral fitting ....................................................... 18

1.7 This thesis ................................................................ 20

2 Background and Methodology ......................................... 23

2.1 History .................................................................... 23
  2.1.1 Non-relativistic spherically symmetric HD wind .......... 24
  2.1.2 Non-relativistic cold cylindrically symmetric MHD wind . 26
  2.1.3 Relativistic cold cylindrically symmetric MHD wind ....... 30
  2.1.4 Non-relativistic warm cylindrically symmetric MHD wind . 31
  2.1.5 Relativistic warm cylindrically symmetric MHD wind .... 32
4.3.3 Self-similarity ........................................... 88
4.4 Discussion .............................................. 91
4.5 Conclusions ............................................. 95
4.A Derivation of the gravity term ......................... 96
4.B Equations for the initial parameter values .......... 97
4.C Gravity in the energy equation ....................... 98

5 Linking accretion flow and particle acceleration in jets. II. Self-similar Jet Models with Full Relativistic MHD Inertia 99
5.1 Introduction .............................................. 100
5.2 Method .................................................. 102
  5.2.1 Background .......................................... 102
  5.2.2 The new gravity term .............................. 103
  5.2.3 Comparison with kinetic gravity term ............ 105
  5.2.4 Effects of the full gravity term .................. 105
  5.2.5 Approach to finding solutions .................... 106
5.3 Results .................................................. 106
  5.3.1 First solution ....................................... 107
  5.3.2 Exploring parameter space ....................... 109
  5.3.3 Self-similarity ..................................... 111
5.4 Discussion and conclusions ............................ 113

6 Discussion and conclusions ................................ 117

7 Samenvatting .............................................. 127

8 Acknowledgments ......................................... 131
Forged in the cauldron of relativistic gravity, with magnetic fields, highly energetic particles, and photons of all energies as their ingredients, stirred by the accretion disc or space itself, jets are a potent concoction. Accelerating particles to very near the velocity of light, heating the ambient medium to thousands of degrees, pushing it around for thousands of parsecs, they are one of the main sculptors of our Universe. While studied for decades, certain aspects are still a mystery. To gain more insight, we have to bridge the big and the small, the brief and the seemingly endless. So let us start by thinking big.

1.1 Active galactic nuclei

Most galaxies are thought to harbour a supermassive black hole (SMBH) in their centre with a mass between $10^5 - 10^{10}$ $M_\odot$ (Kormendy & Richstone 1995; Magorrian et al. 1998). When there is matter flowing towards this BH, the core of the galaxy can turn into a complex engine that emits copious amounts of radiation, sometimes dominating the combined stellar radiation from the host galaxy, and highly relativistic matter. The core of the galaxy, and by extension the galaxy itself, is then called an Active Galactic Nucleus (AGN). The physical processes liberating this energy is far more efficient than nuclear fusion, ranking AGN among the most efficient engines in the Universe.

The prevailing picture of the physical structure of AGN, called the AGN paradigm, is shown in figure 1.1 (Holt et al. 1992). At the centre is a SMBH, attracting the matter surrounding it. Some distance from the centre, clouds of gas slowly moving in this gravitational potential produce narrow emission lines, forming the narrow-line region (NLR), while clouds close to the BH emit strong Doppler-broadened optical and ul-
1 Introduction

Figure 1.1: Suggested structure of an AGN and the effects of orientation on the AGN class observed. The central SMBH is surrounded by an accretion disc, which in turn is embedded in a torus. The region within the torus, where gas clouds are moving very fast, is called the broad line region. This region can be seen directly only from certain angles. Farther from the centre, where the velocity of the clouds is lower, is the narrow line region, which can be observed from all angles. When we are looking along the axis of symmetry, the jet emission dominates the spectrum. Within the unification scheme radio-loud AGN have jets, while radio-quiet AGN do not. Adapted from figure 1 in Urry & Padovani (1995).

Traviolet emission lines, forming the broad-line region (BLR). Surrounding this BLR is a dusty torus, blocking the emission from this region for certain viewing angles. Within this torus, matter is flowing towards the BH, losing its potential gravitational energy, providing enormous amounts of energy, and its angular momentum through viscous or turbulent processes in an accretion disc. Some of the accreted matter is lost to the BH, some may be heated up to be ejected in a disc wind, but another part can be accelerated along helical magnetic field lines produced in the disc or by the spinning BH, forming a relativistic jet. A corona of hot electrons above the accretion disc may form a transitional region between the disc and the jet, but the exact details of jet formation are still the subject of extensive study and debate.
1.1 Effect of AGN on galaxy evolution

Due to the enormous energy liberated in the core of the galaxy, an AGN can have a significant impact on the galaxy surrounding it. The jet can impart kinetic energy to the galactic medium, heating it and possibly pushing it out of the galaxy altogether. Since stars form from cold gas, an AGN can significantly reduce the star formation rate and is therefore intimately connected with the evolution of the galaxy. This connection, called AGN feedback, is exemplified by the fact that the central BH mass is tightly related to galactic properties, such as the velocity dispersion of the galaxy’s bulge (Ferrarese & Merritt 2000; Gebhardt et al. 2000; Tremaine et al. 2002; Gültekin et al. 2009), the bulge luminosity (Dressler 1989; Kormendy & Richstone 1995; Gültekin et al. 2009), and the mass of the bulge (Magorrian et al. 1998; Häring & Rix 2004), although the correlation with the velocity dispersion is the tightest (Beifiori et al. 2012). These relations can be explained best by postulating the BH regulates its own growth (e.g., Tabor & Binney 1993; Ciotti & Ostriker 1997; Hopkins et al. 2009). In simulations AGN feedback can explain the old stellar populations in massive elliptical galaxies, which in hierarchical structure formation should still be forming stars (Ferreras & Silk 2000; Trager et al. 2000; Graves et al. 2009), as well as why there are far fewer very bright and very massive galaxies than expected (Croton et al. 2006). In clusters of galaxies, the intracluster gas is expected to radiatively cool and flow towards the central galaxy increasing the mass of the galaxy and enabling star formation (Fabian 1994). Observations show this does not happen, because AGN jets heat the intracluster medium (Mathews & Brighenti 2003) and create giant cavities in the gas, filled with radio emission (Churazov et al. 2000; Giacintucci et al. 2011, e.g.). Understanding jets thus is essential to galaxy evolution, through the interplay between the galaxy, its BH, and the large scale environment.

1.1.2 AGN classes

Many AGN have been detected and they have been classified according to their observed properties. Although AGN appear in a myriad of classes, two main types of AGN have been distinguished: Type 1 AGN have bright continua and broad emission lines from high-velocity gas in their spectra, while Type 2 AGN have weak continua and only narrow emission lines. In addition there is a small number of AGN with atypical spectral characteristics. Apart from differences in their spectral lines, AGN also cover a range of luminosities, especially in their radio emission. This has led to the adjectives radio-loud, when the ratio of the radio flux at 5 GHz to the optical flux in the B-band $F_5 / F_B \gtrsim 10$, and radio-quiet when this ratio is smaller.

Type 1 AGN that are radio-quiet include the low-luminosity Seyfert 1 galaxies with comparatively very bright nuclei, which can only be detected at small cos-
mological distances. At higher luminosity are the radio-quiet quasars (QSOs; from quasi-stellar objects), which are relatively rare in the local universe, with the most luminous quasars inhabiting the most massive galaxies. Radio-loud AGN of Type 1 are called Broad-Line Radio Galaxies (BLRGs) at low overall luminosities, and radio-loud quasars at higher luminosity.

For the Type 2 AGN the radio-quiet class is called Seyfert 2 or Narrow Emission Line Galaxies (NELGs) and the radio-loud class the Narrow-Line Radio Galaxies (NLRGs). The radio-loud AGN have been divided by Fanaroff & Riley (1974) in two classes with very distinct morphologies. Although the original distinction was made on whether the distance between the regions of highest brightness is less (FR1) or more (FR2) than half of the total extent of the source, in practice FR1 sources have intensities decreasing from the centre, while FR2 sources have two clear hot-spots in the prominent radio lobes surrounding the galaxy.

A minority of AGN have atypical spectra, characterised by rapid variability and no strong emission lines. This group includes the radio-loud blazars, combining the low-luminosity BL Lacertae (BL Lac) objects and the high-luminosity flat-spectrum radio quasars (FSRQ), and optically violent variables (OVVs).

1.1.3 Unification through orientation

In order to reduce this myriad of classes, people have tried to unify them in a single model. One unification scheme based on the orientation of the galaxy with respect to our line of sight, was developed by Urry & Padovani (1995), relating specific spectral properties to different regions of the AGN (see figure 1.1). In this scheme the distinction between the radio-loud and radio-quiet AGN is caused by the presence or lack of a jet, which emits mostly, although not exclusively, in radio.

The distinction between Type 1 and Type 2 AGN is more subtle. The broad lines in the spectrum vary, which suggests that the region emitting this radiation is small. The matter close to the BH rotates very quickly, causing broad lines in the spectrum. These two arguments lead to a proposed small region surrounding the BH that provides the observed broad lines. The narrow lines in the spectrum do not vary significantly, suggesting the emitting region is extended, with the narrow lines advocating a region far away from the BH, where the velocities are lower. If there were an optically thick torus surrounding the broad line region (BLR), for certain viewing angles this BLR would be obscured by the torus. Starting from an edge-on perspective, we would only see narrow lines and radio emission if there were a jet (Seyfert 2 and NLRG). Then as we go towards a face-on view, first the BLR would come into view, dominating the emission from the narrow line region (NLR), changing the perceived AGN to a Seyfert 1 and a BLRG. When the central region of the AGN comes into full view, the luminosity increases and we see either a radio-
quiet or radio-loud QSO. When the viewing angle is smaller than the jet opening angle, we see the quickly varying BL Lac and OVV objects.

### 1.1.4 Intrinsic differences

While the unification scheme of Urry & Padovani (1995) provides explanations for most observed features of AGN, there are still a few that cannot be explained within this scheme. By comparing BL Lac objects with a model including emission from the jet, host galaxy, and torus Plotkin et al. (2012a) found that, as opposed to FSRQs, by leaving out the torus the BL Lac objects were fitted much better. This result seems to indicate that in these sources the postulated torus is missing, possibly as a result of a lower accretion rate. The presence or absence of a jet may either be caused by the surrounding medium, which could for example explain the differences between FR1 and FR2 sources, or by intrinsic differences such as the accretion rate, the mass, or the spin of the BH, which could have an effect on the power of the jets in AGN (Garofalo et al. 2010).

While there is evidence of intrinsic changes within AGN, unfortunately, due to the enormous masses and length scales involved, these changes can take many thousands to millions of years. Since general relativity predicts that BH physics scales with mass, in order to see these changes on human time scales, we have to go to smaller systems.

### 1.2 Black hole X-ray binaries

Stellar mass BHs are ∼ 10^4–10^10 times as small as their supermassive counterparts. When accreting from a companion star, they emit large amounts of X-rays. Since they were first noticed in X-ray observations, these systems are called black hole X-ray binaries (BHXRBs). As in AGN, due to angular momentum conservation the accreted material forms an accretion disc, which is heated up and magnifies magnetic fields by viscous processes such as the magnetorotational instability (Balbus & Hawley 1991). These magnetic field lines can guide material from the disc away from the BH, either via a disc wind or a collimated jet (see figure 1.2).

#### 1.2.1 Accretion states

Most BHXRBs are variable, going through spectral changes and eventually returning to its original spectrum in a period of months to a year. These cycles are best seen in a hardness-intensity diagram (HID), which plots the X-ray luminosity versus the slope of the X-ray spectrum (Fender et al. 2004). Figure 1.3 shows the HID for GX 339-4, a galactic BHXRB. Starting from the lower right, after a period of low luminosity,
the total X-ray luminosity increases, causing the source to move up. Next the hard X-ray luminosity drops, while the soft X-ray luminosity still increases, leading to a softening of the spectrum, and a shift to the left. After the source spends some time overall X-ray luminosity goes down again, the spectrum returns to its original shape.

These different spectra are thought to be tied to specific configurations of the system, so-called accretion states. Both the mass accretion rate and the geometry of the disc changes as we go up in luminosity. As the mass accretion rate increases, the inner radius of the disc decreases, and from a geometrically thick geometry, the disc becomes geometrically thin. It is believed a thin disc cannot support strong magnetic fields (Meier 2001) and hence the jet and the corresponding radio emission are eventually quenched.
1.3 Mapping of AGN classes onto BHXRB accretion states

But is it really possible to use the knowledge gained from BHXRBs and apply them to SMBHs? There is some evidence this is actually the case. Aside from obvious similarities in the appearance of accreting stellar-mass and supermassive BHs, giving microquasars their name, a correlation has been found between the compact radio luminosity, assumed to be jet emission, the X-ray luminosity, assumed to come from the disc, and BH mass for both stellar mass and SMBHs, called the fundamental plane of black hole accretion/activity (Merloni et al. 2003; Falcke et al. 2004). This correlation is particularly strong when BHXRBs in the jet-dominated hard state are compared with their supermassive counterparts, the low-luminosity AGN (LLAGN). As Merloni et al. (2003) also included BHXRBs in other states and high-luminosity quasars for their analysis, their correlation had more scatter. Recently Plotkin et al. (2012b) were able to derive a more refined relation: \( \log L_X = (1.45 \pm 0.04) \log L_R - (0.88 \pm 0.06) \log M_{\text{BH}} - 6.07 \pm 1.10. \) While there is substantial scatter in the relation,
it does represent a correlation that holds over eight orders of magnitude in mass (see figure 1.4).

So if, as general relativity predicts, there is no fundamental difference between BHs of different masses, it is possible AGN go through the same accretion states as BHXRBs, with radio-loud quasars corresponding to the hard state and radio-quiet to the soft state. As mentioned before, due to the much longer time-scales it is impossible to observe a state change in an AGN on human time-scales, but by looking at the remnants of jet activity it may be possible to say something about the past behaviour of AGN. In clusters of galaxies, when the central AGN jet is active it can blow a bubble in the intracluster gas (see figure 1.5). By making assumptions about the geometry and kinetics of these bubbles, it is possible to estimate the age of the cavities. For Hydra A this analysis gives ages of around $10^8$ years for both the duration and the time between cavities Wise et al. (2007). With a central BH mass of $9 \times 10^8 M_\odot$ (Rafferty et al. 2006), these time-scales would be of the order of a year for stellar-mass BHs, corresponding roughly to the observed accretion states cycle in BHXRBs.

Although the mapping is not yet complete, Körding et al. (2006) plotted almost 5000 quasars on disc-fraction/luminosity diagram (DFLD), a generalisation of the HID for BHXRBs. As the frequency of the radiation depends on the mass of the black hole, and the range of black hole masses is rather wide, the DFLD has the disc luminosity plus the power-law component luminosity ($L_D + L_{PL}$), as a measure of

**Figure 1.4:** Edge-on view of the correlation between the compact radio luminosity, the X-ray luminosity, and BH mass for both stellar mass and SMBHs, including high-energy cutoff BL Lacs (HBLs) from the Sloan Digital Sky Survey (SDSS). The solid line shows the best fitting function. Since beaming can have a significant effect on BL Lacs, they have been debeamed with an assumed Doppler factor of 7 (Ghisellini et al. 1993). Figure adapted from Plotkin et al. (2012b).
luminosity, plotted versus the power-law luminosity divided by the sum \([L_{PL} / (L_D + L_{PL})]\), as a measure of hardness. The result has some similarities to the HID for BHXRBs, but only in a statistical sense. The hope is that in the future it may be possible to put an individual AGN on an HID analogue. Despite the fact stellar-mass BH systems appear \(10^5\) times smaller than AGN\(^1\), they are understood in more detail due to their variability. Extending the knowledge gained from BHXRBs to SMBHs would be a great step towards being able to predict the activity of these active galactic nuclei and their effects on galaxy formation, evolution and ultimately the large-scale structure of the Universe. Conversely, the much higher spatial resolution of AGN gives us information unavailable in BHXRBs, so the combination promises to shed light on jet formation. Since the jet is a fundamental part of the system and contributes significantly to the spectrum, it is vital we understand the effects it has. It

\(^1\)V616 Monocerotis, one of the closest BHXRBs, has a BH mass \(M_{BH} = 6.6 \pm 0.25 M_\odot\) and distance \(d = 1.06 \pm 0.12\) kpc (Cantrell et al. 2010) for an angular size of the Schwarzschild radius of \(1.2 \times 10^{-10}\) arcseconds, while for the supermassive BH at the centre of our Galaxy, Sagittarius A* (Sgr A*) with \(M_{BH} = (4.3 \pm 0.38) \times 10^6 M_\odot\) and distance \(d = 8.3 \pm 0.35\) kpc (Gillessen et al. 2009) the angular size of the Schwarzschild radius is \(1.0 \times 10^{-5}\) arcseconds.
1 Introduction

also seems to regulate other parts of the system, increasing its importance.

1.4 Accretion discs

If we want to be able to explain the observations of BHXRBs, we need to model
the hydrodynamics and radiation processes of gas in orbit around the BH. The best-
known one is the geometrically thin, optically thick disc model developed by Shakura
& Sunyaev (1973). Matter falling towards the BH, either from a companion star,
or from the interstellar medium, usually possesses angular momentum with respect
to the BH. As it gets closer to the BH the matter starts to rotate around it. The
differential rotation due to the Keplerian orbits causes shearing between the different
radii. Viscous processes facilitate angular momentum exchange, allowing the matter
to spread out in a disc, and also heat up the disc, getting hotter as it moves deeper
into the potential well. Due to the temperature of the disc, which can reach \( \sim 10^7 \) K
near the BH, the matter is in a plasma state, radiating as a black body.

It is not clear which physical processes are responsible for the viscosity. One pos-
sibility is turbulence, but since hydrodynamic instabilities are not sufficient (Balbus
& Hawley 1998), it seems more likely the turbulence is supported by magnetic fields
(McKinney & Gammie 2002) via the magnetorotational instability (Balbus & Haw-
ley 1991). When a field line threading the disc develops a small radial kink, the field
line will be stretched due to the differential rotation. This causes the inner material
to slow down, falling to a lower orbit, while the outer material is sped up, reaching a
higher orbit. In this way angular momentum is exchanged between different annuli.
Only if angular momentum can be transported outwards, can matter eventually fall
into the BH.

The origin of the magnetic fields is a puzzle. Since BH do not have a material
surface, they cannot support a magnetic field. While in theory a charged BH could
solve this problem, the electric force would dominate the gravitational force, and
they would preferentially accrete matter with opposite charge, becoming effectiv-
ely neutral within a few light-crossing times. There are several other methods by which
magnetic fields can appear in the disc. The accreting matter can drag weak magnetic
fields with it into a smaller volume, which would increase the field strength. This
initial magnetic field could be stretched azimuthally by the MRI (Balbus & Hawley
1991). Since the disc gas and the magnetic field remain in pressure balance, a strong
magnetic field has a corresponding lower matter density, making it rise buoyantly
(Parker 1966). This vertical movement strengthens the vertical field, which can then
feed into the MRI (Tout & Pringle 1992). Yet by ignoring vertical gravity, Hawley
et al. (1995) also found some dynamo effects, without the Parker instability. While
progress is being made (Johansen & Levin 2008), it is clear that exactly how magnetic
fields are generated is still one of the main issues in accretion physics.

Close to the BH there are no stable orbits, so the thin disc has to terminate. This happens at the innermost stable circular orbit (ISCO), which is located at $r_{\text{ISCO}} = 6 \, r_g$ for a Schwarzschild BH with an event horizon $r_S = 2 \, r_g$, and coincides with the event horizon of a maximally-rotating (or extreme) Kerr BH $r_{\text{ISCO}} = r_K = r_g$. When the matter reaches the ISCO, it will have radiated $0.057 \, mc^2$ for a Schwarzschild BH and $0.42 \, mc^2$ for an extreme Kerr BH. The potential energy liberated by viscous processes is radiated locally as a black body, but since the temperature of the plasma increases as the matter gets closer to the BH, instead of a single black body spectrum, the disc radiates as a superposition of many black bodies of different temperature. The standard multi-colour disc (MCD) model used for this does not take the torque-free boundary at the ISCO into account and therefore emits too much radiation there (Gierliński et al. 1999). Magnetic fields can couple the disc to a Kerr BH, extracting rotational energy, which can also significantly change the spectrum of the inner accretion disc (Wilms et al. 2001).

The above is a description of the high/soft state, which has a dominant thermal component in its spectrum believed to come from the disc, and occurs at high sub-Eddington accretion rates. In contrast, the low/hard state occurs at lower accretion rates, and is dominated by a non-thermal power-law component. One possible source for this component is the inverse Compton process. Low-energy seed photons, either from the disc or the jet, can gain energy from collisions with hot electrons in a corona surrounding the disc. This process causes a hump at higher energies, which can resemble a power law. Another source could be emission from the jet.

For sources at very low accretion rates, the accretion rate determined from the spectrum by assuming an MCD is much lower for the inner disc than the outer disc (McClintock et al. 1995). While originally explained by matter piling up due to a disc instability, later it was noticed a more realistic solution was supposing the thin disc is truncated before the ISCO (Narayan et al. 1996), with the interior being filled by an advection-dominated accretion flow or ADAF (Narayan & Yi 1994, 1995b). This ADAF radiates away only a fraction of the available gravitational energy, with the remainder being advected into the BH, alleviating the very strict upper limits on the accretion rate from the spectrum. Since the flow is not cooled radiatively, the electron temperature can reach $10^9$–$10^{10}$ K, which causes a puffed up, possibly nearly spherical, flow geometry (Narayan & Yi 1995a). ADAFs have been used to fit both stellar mass BHs (Narayan et al. 1996) and LLAGN (Narayan et al. 1995; Lasota et al. 1996).

Extensions to the original ADAF model include the advection-dominated inflow-outflow solutions (ADIOS), where the energy stored in the flow is used to drive away part of the accreting matter in the form of a wind (Blandford & Begelman 1999),
1 Introduction

Figure 1.6: Schematic of the thin disc (horizontal bars) and ADAF (dots) in different accretion states as a function of the Eddington-scaled mass accretion rate $\dot{m}$. Although the very high state is shown, it is not part of the unification scheme (Esin et al. 1997).

the convection-dominated accretion flow (CDAF), where the flow becomes convectively unstable due to a low viscosity (Igumenshchev & Abramowicz 1999; Narayan et al. 2000; Quataert & Gruzinov 2000), and the magnetically-dominated accretion flow (MDAF), which is situated within an ADAF and is supported by well-ordered magnetic field (Meier 2005). Collectively these models are known as radiatively inefficient accretion flows (RIAFs).

One possible picture of how the disc relates to the accretion states has been sketched by Esin et al. (1997, see figure 1.6). At very low accretion rates, the disc is truncated at a large radius and the ADAF has a low density. As the accretion rate increases, the density of the ADAF first increases as well, but then, as the disc starts to move inwards to the ISCO, decreases again. While successful in describing the spectral evolution of BHXRBs, the model does not unify the very high state, explain flaring events, or describe in detail the transition from a cold disc to a hot ADAF. It is also clear the different accretion states depend on more parameters than only the
1.5 Jets

It is possible to avoid the requirement of RIAFs altogether. Merloni & Fabian (2001) showed that the corona has to be strongly magnetised in order to explain the observed hard X-ray, and acts as a magnetic reservoir intimately connected to the accretion disc. This reservoir contains enough energy to power the high-energy emission. This hot, magnetically-dominated corona is an ideal site for launching jets, which, if radiatively inefficient, would make the source overall radiatively inefficient, without the need for a RIAF (Merloni & Fabian 2002).

1.5 Jets

Astrophysical jets are collimated outflows with a roughly helical magnetic field structure. These jets have been observed around as varied objects as young stellar objects, white dwarfs, neutron stars, and black holes (BHs), and they are thought to play a key role in the most energetic events of our Universe, the gamma-ray bursts. While formed in very different environments, it is believed jets need three basic ingredients: a source of power (either matter accreting onto a compact object, or the spinning object itself), rotation, and magnetic fields. Because they transfer matter from close to the BH to potentially large radii, jets can be an efficient method of transporting angular momentum, diminishing the need for viscous processes in the accretion disc. They also deposit a large amount of energy and momentum in the ambient medium, heating and displacing it. In the case of AGN, this can affect the evolution of their host galaxy (e.g. Best et al. 2006).

The matter content of jets is still mostly unknown. It is possible they consist of electron/positron pairs, a proton/electron mix, or a combination of the two. It seems clear, however, that the electrons (and possibly positrons) are responsible for most of the radiation, through the synchrotron process, where relativistic electrons circle around field lines emitting polarised radiation, and the inverse-Compton process, where photons gain energy through elastic collisions with high-energy electrons.

1.5.1 Observations of jets

Regardless of the size of the compact object, many jets can accelerate particles to highly relativistic speeds. The Lorentz factors of AGN jets are found to be usually \( \lesssim 10 \), although some rare sources may go up to 50 (Lister et al. 2009). For BHXRBs the typical jet Lorentz factors are \( \lesssim 2 \), with GRS 1915+105 seemingly occasionally reaching up to 5 (Mirabel & Rodríguez 1999).

It is not yet clear whether the spin of the BH has an effect on the jet power. For AGN it is a theoretical possibility with some observational justification (Garofalo et al. 2010). For BHXRBs the observational evidence is far less certain. Fender et al.
(2010) posit no clear relation between these two quantities exists, while Narayan & McClintock (2012) claim there is one. The latter authors argue that in the hard state the steady jet is produced sufficiently far away from the BH that its spin has no appreciable effect. Conversely, during the transition from the hard to the soft state the inner edge of the accretion disc reaches the ISCO, resulting in a shock or other instability intermittently launching blobs of plasma at higher relativistic velocities (Fender et al. 2004). Since the radius of the ISCO depends on the spin of the BH, this would explain a correlation between spin and jet power. However, with the current issues of reliably determining the spin, the lack of a consistent definition of the jet power, and only four objects plus one lower limit, it seems too early to assert its validity. But if the relation is confirmed, it would provide a valuable method to ascertain one of the most difficult parameters from the broadband spectrum.

The observational characteristic of jets is the synchrotron emission, observed in both AGN (Marscher & Gear 1985) and BHXRBs (Fender 2002). This emission is produced by relativistic electrons rotating around the magnetic field lines. The emission we observe is usually appreciably polarised, suggesting the magnetic field lines are well-ordered. The resulting spectrum depends on the underlying electron energy distribution. For a thermal distribution, expected at the base of the jet, the spectrum has an exponential cutoff at high frequencies. A flat or slightly inverted radio spectrum is seen in the hard state in BHXRBs (Fender 2001), as well as in LLAGN (Ho 1999). This can be interpreted as a superposition of synchrotron spectra from a population of electron with a power-law energy distribution $dn = N_0 E^{-p} dE$ (Blandford & Konigl 1979). With this distribution the spectrum at a location in the jet with a certain density is $F_\nu \propto \nu^{5/2}$ for frequencies $\nu$ where the jet is optically thick, and $F_\nu \propto \nu^{-(p-1)/2}$ for frequencies where the jet is optically thin. Since the density and magnetic field strength of the jet decrease outwards, also the peak of the spectrum and the total power decrease. All these synchrotron components along the jet add up to a nearly flat radio spectrum (see figure 1.7).

This flat spectrum extends to the frequency corresponding to the peak of the population of electrons with a power-law energy distribution closest to the BH. Beyond this frequency the spectrum falls as the above mentioned $F_\nu \propto \nu^{-(p-1)/2}$. In AGN jets this break typically occurs in the GHz range (Ho 1999), and for BHXRBs this break is predicted to occur in the infrared (Markoff et al. 2001, 2003; Heinz & Sunyaev 2003). The observed slope of the optically thin component corresponds to a power-law index $p \sim 2–2.6$, which means, depending on the exact break frequency, the optically thin tail of this component can extend well into the X-rays. If the acceleration region is located rather close to the BH, the optically thin tail can be the dominant contribution in the X-ray, diminishing or removing the need for an inverse-Compton, or disc component. Determining the break frequency can thus tell us which process provides...
the X-ray flux, and consequently what the conditions in the jet and corona are. A picture is evolving that in the hard state, at low accretion rates, the X-ray emission is predominantly optically thin synchrotron from the jet, while at higher accretion rates, in the soft state, emission from the inner accretion disc is the main contributor. However, this view has to be corroborated by different approaches.

Although for BHXRBs the band where the break occurs, is usually dominated by the spectrum of a stellar companion or the accretion disc, the break has been observed in GX 339-4 during the hard state (see figure 1.8; Corbel & Fender 2002; Gandhi et al. 2011). It corresponds to the region in the jet where electrons first get accelerated from a thermal into a power-law energy distribution, with a higher break frequency indicating a region closer to the BH. From fitting multiple spectra of BHBs and AGN this region seems to be offset from the BH, with the height in the range of \( \sim 10-1000 \, r_g \), where \( r_g \) is the gravitational radius, \( GM/c^2 \), with \( G \) the gravitational constant, \( M \) the mass of the BH, and \( c \) the velocity of light (Markoff et al. 2001, 2003, 2005; Migliari et al. 2007; Gallo et al. 2007; Markoff et al. 2008; Maitra et al. 2009a).

This height should also cause the synchrotron radio core to be offset from the BH in direct imaging. Although the required spatial resolution is very high, for nearby
1 Introduction

Figure 1.8: Broadband radio–X-ray spectrum of two hard states in GX 339-4 observed in 1981 (filled symbols) and 1997 (open symbols). The long-dashed and short-dashed lines correspond to the optically-thick and optically thin regime, respectively, with spectral indices $+0.15$ and $-0.6$ for 1981 and $+0.08$ and $-0.65$ for 1997. Since an extrapolation of the X-rays intersects the near-infrared data with a slope compatible with optically thin synchrotron, it is very possible the X-rays are emitted by the jet. This interpretation is corroborated by the fact the whole spectrum is lower by a factor of four in the 1997 data, suggesting a common physical origin for all three bands. The optical data of 1981 are consistent with thermal emission, presumably from an accretion disc. Taken from Corbel & Fender (2002).

AGN this would be possible. Indeed in M87, an AGN with a large angular diameter at a distance of $17.0 \pm 0.3$ Mpc (Tonry et al. 2001) with a large SMBH of $6.4 \pm 0.5 \times 10^9 M_\odot$ (Gebhardt & Thomas 2009, although note this value is twice the mass found in previous studies), the offset seems to be $\sim 100 r_g$ (Junor et al. 1999; Walker et al. 2008), in line with the values found for other sources. The offset of the radio core with respect to the BH has been extremely stable over the last years (Asada et al. 2011).

1.5.2 Theoretical models of jets

Since the exact formation of jets is still unclear, several models have been proposed that provide the energy in different ways. The Blandford-Znajek model (Blandford
& Znajek 1977) describes field lines threading a spinning BH, extracting rotational energy to power a magnetically dominated jet populated with pair-produced electrons and positrons. The Blandford-Payne model (Blandford & Payne 1982) on the other hand, has the jet anchored in the accretion disc, with the rotation providing a centrifugal force accelerating the matter along the field lines. A later extension to the Blandford-Payne model describes a relativistic jet where the initial acceleration is provided by the thermal energy of the matter, which is later taken over by magnetic acceleration (Vlahakis & Königl 2003a). It is possible both types exist simultaneously in the same source, with the Blandford-Znajek jet forming the spine, and a Blandford-Payne jet surrounding it as a sheath. For an overview of the development of jet models, we refer the reader to chapter 2.

The fundamental plane of black hole accretion, mentioned above, is a natural consequence if accretion processes and jets scale with the gravitational radius, the fundamental length in the system (Falcke & Biermann 1995; Markoff et al. 2003; Heinz & Sunyaev 2003). Here we will summarise the derivation of this relation given by Heinz & Sunyaev (2003). If jet and accretion processes are indeed scale free, we can separate variables in mass and radius and write all dynamically relevant quantities as

\[ f(M, \dot{m}, a, r) = \phi_f(M, \dot{m}, a) \psi_f(r/r_g, \dot{m}, a), \]

where \( \dot{m} \) is the Eddington-scaled accretion rate, and \( a \) is the spin of the BH. We observe jets in the hard state, which is well described by a Shakura-Sunyaev disc, with \( B \propto M^{-1/2} \). The jet emits synchrotron radiation from a power-law distribution of electrons:

\[ dn = N_0 E^{-p} dE, \]

with \( p \) the power-law index, which observations tell us is close to \( p = 2 \). Assuming equipartition gives us \( N_0 \propto B^2 \propto M^{-1} \). As there is not too much difference between the Lorentz factors of BHXRBs and AGN, we assume there is no mass dependence, which allows us to combine the dependence on viewing angle due to Doppler beaming and optical depth effects into a function \( \zeta(\theta) \) independent of mass. Next we calculate the surface integral over the jet surface brightness

\[ S_\nu \sim \zeta(\theta) j_\nu (1 - e^{-\tau_\nu})/\alpha_\nu, \]

where \( j_\nu \) is the synchrotron emissivity, \( \tau_\nu \) is the optical depth, and \( \alpha_\nu \) is the synchrotron self-absorption coefficient. We define the spectral index \( \alpha \equiv -\partial \log(F_\nu) / \partial \log(\nu) \). Now we can obtain an expression for

\[ \partial \log(F_\nu) / \partial \log(M) = \xi_M \]

by substituting in \( \alpha \) and using the dependencies of \( B \) and \( N_0 \) on \( M \):

\[ \xi_M = \frac{2p + 13 + 2\alpha}{p + 4} + \frac{\partial \log(\phi_B)}{\partial \log(M)} \left( \frac{2p + 3 + \alpha(p + 2)}{p + 4} \right) + \frac{\partial \log(N_0)}{\partial \log(M)} \left( \frac{5 + 2\alpha}{p + 4} \right) \approx \frac{17}{12} - \frac{\alpha}{3}, \]

for our assumed values. This equation represents the exponent of the relation between the jet radio emission and the BH mass. Following the same steps, we can derive
an equation for the exponent of the relation between the jet radio emission and the accretion rate:

$$\xi_{\dot{m}} = \frac{\partial \log(\phi_B)}{\partial \log(\dot{m})} \left[ \frac{2p + 3 + \alpha(p + 2)}{p + 4} \right] + \frac{\partial \log(\phi_{N_0})}{\partial \log(\dot{m})} \left( \frac{5 + 2\alpha}{p + 4} \right) \approx \frac{17}{12} - \frac{\alpha}{3}, \quad (1.2)$$

for ADAF-type accretion with $N_0 \propto B^2 \propto \dot{m}$. Now following Markoff et al. (2003), we can use the equations:

$$\log L_R = \xi_{\dot{m}} \log M + \xi_{\dot{m}} \log \dot{m} + K_1, \quad (1.3)$$

$$\log L_X = \log M + q \log \dot{m} + K_2, \quad (1.4)$$

to cast everything into the form:

$$\log L_R = \xi_{RX} \log L_R + \xi_{RM} \log M_{BH} + b_R, \quad (1.5)$$

$$\log L_X = \xi_{XR} \log L_R + \xi_{XM} \log M_{BH} + b_X. \quad (1.6)$$

If we assume a flat radio spectrum ($\alpha = 0$), and $q \approx 2$, close to an ADAF, we obtain:

$$\xi_{RX} = \frac{\xi_{\dot{m}}}{q} \approx 0.71, \quad (1.7)$$

$$\xi_{RM} = \xi_{\dot{m}} - \frac{\xi_{\dot{m}}}{q} \approx 0.71, \quad (1.8)$$

$$\xi_{XR} = \frac{q}{\xi_{\dot{m}}} \approx 1.4, \quad (1.9)$$

$$\xi_{XM} = 1 - \frac{q}{\xi_{\dot{m}}} \xi_{\dot{m}} \approx -1. \quad (1.10)$$

not too far off the observed values. The strength of this analysis is that it is independent of the assumed jet model. By observing the different sources, we can determine the exponents of the fundamental plane correlation, as well as $\alpha$ and $p$, and in turn determine the dependence of the magnetic field and power-law distribution on the mass and accretion rate, and by constraining $q$, whether it is radiation or gas pressure supporting the disc, or if it is actually an ADAF. In this way the fundamental plane provides a very strong check on the origin of the radiation.

### 1.6 Spectral fitting

A spectral fitting code was developed to test the hypothesis that the hot magnetised corona could be the base of the jet, also called the jet-disc symbiosis. This program describes a standard thin, optically thick accretion disc (Shakura & Sunyaev 1973), that truncates at a certain radius and turns into a hot, radiative-inefficient accretion
1.6 Spectral fitting

The disc radiates as a multi-temperature black body, contributing to the infrared through ultraviolet, depending on the size and temperature of the disc. A certain fraction of the accretion energy is powering the jet, shared between the magnetic field and the bulk kinetic energy of the particles with a certain ratio, with the protons carrying the kinetic energy and the electrons causing thermal synchrotron and inverse Comptonisation of disc and jet photons. The jet then exits a nozzle with the proper sound speed \((\gamma_s \beta_s c \approx 0.4c)\) for a hot electron/proton plasma) and expands at the same rate. As the jet expands, it cools adiabatically, causing a thermal pressure, which accelerates the jet. This weak acceleration provides the only collimation. At a certain height, the particles, which are assumed to have a quasi-thermal distribution, are accelerated into a power-law distribution, changing the local synchrotron spectrum. These synchrotron components add up along the jet to form a flat (Blandford & Konigl 1979), or slightly inverted spectrum, covering the radio to possibly the X-ray, depending on the height of the acceleration region. The particles have to be continuously accelerated beyond the start of the acceleration region, since otherwise the radiative losses would quickly cool down the electrons again, contrary to observation. For an XRB there can be an additional companion star radiating as a black body, while for an AGN there can be an additional iron line. The total spectrum is thus built up from a multi-temperature black body, a pre-shock and post-shock synchrotron component, an inverse Compton component and an optional black body. An irradiated disc component has also been added (Maitra et al. 2009a). By fitting the spectrum of an observed source, parameters such as the inner radius and temperature of the accretion disc can be determined.

This code has been expanded and refined over the years (Falcke & Biermann 1995; Falcke 1996; Falcke & Markoff 2000; Markoff et al. 2001, 2003, 2005, 2008; Maitra et al. 2009a), and has been used to model both AGN (Falcke 1996; Falcke & Markoff 2000; Markoff et al. 2008) and XRBs (Markoff et al. 2001, 2003, 2005; Maitra et al. 2009a). With the exception of Sgr A∗, in all these sources there is a remarkable similarity in fitted parameters, with for example the height of the acceleration region lying in the narrow range of \(10r_g - 400r_g\) (Markoff et al. 2008; Maitra et al. 2011).

However, since it is built upon an HD model (Falcke & Biermann 1995; Falcke & Markoff 2000), the magnetic field simply enters as a global parameter, serving to guide the flow and enable synchrotron radiation, but not to accelerate and collimate the material. Due to its HD nature, the model has merely weak longitudinal acceleration due to thermal pressure and can therefore only describe systems where the jets are neither highly accelerated, nor highly collimated.

Incorporating an MHD model has several advantages. First, the model will treat the magnetic fields in a self-consistent way, with the fields providing the acceleration
1. Introduction

and geometry of the jet. Second, the model will be suited to a wide range of environments, where either the velocity, the magnetic field, the gravitational potential, or all can attain relativistic energies. This inclusion will therefore allow the code to be applied to an even greater variety of sources of both mass scales, and help shed light on the mapping of the BHXRB states to the AGN classes. Since discs provide the boundary conditions for a jet, and our model relates those conditions near the BH to those far away in the jet, by observing the properties of jets in actual BHXRBs, we will be able to constrain certain disc models. Third, it will allow us to independently calculate the power of the jets, which is an essential parameter for AGN feedback models. Fourth, the height of the acceleration region becomes a derived quantity and is removed as a free parameter.

A natural way to accelerate particles into a power-law distribution is via diffusive shock acceleration (e.g. Bell 1978; Drury 1983). Since the highly energetic electrons cool very fast via radiative losses, this shock acceleration must happen everywhere beyond this acceleration point, since we do not observe a decaying power-law distribution anywhere along the jets (e.g. Jester et al. 2001). Since the particles should be accelerated continuously, this shock should be a steady feature within the jet.

Since the height where acceleration starts seems to be relatively similar in several systems, it may be possible to identify this location with a critical point in a magnetohydrodynamical (MHD) flow, especially the magnetosonic modified fast point (MFP) (Blandford & Payne 1982). At the MFP the jet begins to overcollimate, meaning the jet radius decreases again, which could cause recollimation shocks. Another feature of the MFP is that it is the place where the outward flowing jet becomes causally disconnected from the upstream flow, so shocks can occur without disrupting the flow upstream. Such shocks thus could be a stable feature in the jet, continuously accelerating the particles into a power-law distribution. Since the shock is tied to the location of the MFP, we will identify the MFP as the start of the particle acceleration region.

1.7 This thesis

The goal of the research in this thesis is to develop a relativistic jet acceleration model that includes gravity. This jet model can then be used to determine the location of the start of the acceleration region, identified with the height of the MFP, and relate this height to the conditions close to the BH. We want to cross all three singular points in order to have the most reliable physical link between the observed regions and the jet region near the BH. This jet model can then be used to determine physical parameters of black hole systems, based on their broadband spectra.

In the next chapter we give a historical introduction to the field of outflow models, explain the nomenclature used throughout this thesis, as well as an overview of the
steps taken to derive the jet model.

In chapter 3 we present, for the first time, solutions to the relativistic MHD equations that cross the MFP.

In chapter 4 we describe how we include the gravitational force due to kinetic inertia of the plasma into the model. This addition allowed us to find solutions crossing all three singular points. We also show how the properties of the solutions depend on the parameters chosen.

Chapter 5 gives the extension of the model including the gravitational force due to the full relativistic inertia (kinetic, thermal, and electromagnetic), as well as a parameter study of the models.

In chapter 6 we will discuss our results and present our conclusions.
The model developed in this work belongs to the class of axisymmetric, radially self-similar wind models. Every wind model is fully specified by the so-called wind equation. By integrating this wind equation, with possible several supplementary equations, we can obtain a full description of the density, velocity and temperature of the material and, if present, the magnetic and electric field configuration and it is therefore of primary importance to this thesis. In this chapter we will describe this class of models in detail and introduce the important concepts associated with them. We also will give an overview of the steps taken to extend the model to include gravity. Since the basics of the models are the same, we will give only a cursory list of steps in the following chapters.

2.1 History

There has been a long history of wind\(^1\) models, from spherically symmetric hydrodynamic, to axisymmetric magnetohydrodynamic ones. They all share elements in their derivation and also have several concepts in common. By giving a chronological overview, from relatively simple to more advanced, we will introduce the important concepts and show how the different assumptions at the basis of them influence their properties.

\(^1\)While the word wind originally denoted outflowing plasma from a star, the models have been generalised to such an extent that now in the most general sense it can be an (un)collimated outflow or inflow, gravitationally bound or unbound, with velocities ranging from subsonic to relativistic, consisting of neutral matter or a plasma, anchored on any surface, such as a star or a disc.
2 Background and Methodology

2.1.1 Non-relativistic spherically symmetric HD wind

The first wind model was developed in the 1950’s by Parker (1958). By observing the direction of comet tails, Biermann (1951) proposed that the interplanetary gas was moving away from the Sun, which led Parker (1958) to consider the spherically symmetric case. Due to the fact that only the square of the velocity enters the equations, this non-relativistic hydrodynamic model actually describes both a stellar wind and accretion onto a star. While it is relatively simple, it already incorporates some general concepts, which are clear to see from the governing equation. The equations this model and all others are based on are the continuity equation, describing conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0,$$

where \( \rho \) is the matter density, \( t \) is time, and \( \mathbf{V} \) is the velocity vector; the Euler equation, describing the conservation of momentum:

$$\rho \frac{\partial \mathbf{V}}{\partial t} + \rho \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla P + \mathbf{f},$$

where \( P \) is the gas pressure, and \( \mathbf{f} \) is the external force density, which in the case of the Parker wind is the gravitational force density, \( \mathbf{f}_g = -GM\rho/r^2 \), where \( G \) is the gravitational constant, \( M \) is the mass of the star, and \( r \) is the spherical radius; the ideal gas law, which relates the temperature, pressure and density of the gas:

$$P = \frac{\rho kT}{\mu m_H},$$

where \( k \) is Boltzmann’s constant, \( T \) is the gas temperature, \( m_H \) is the mass of a hydrogen atom, and \( \mu \) is the mean molecular weight of the gas in units of \( m_H \); and finally a polytropic energy equation, relating the gas pressure and density:

$$P = Q\rho^\Gamma,$$

where \( Q \) is the adiabat, and \( \Gamma \) is the polytropic index. The assumptions going into the model are time independence and spherical symmetry \((r, \theta, \phi)\). Combining the above mentioned equations, and taking all the assumptions into account, it is possible to derive an equation for the derivative of the velocity squared with respect to the radius:

$$\frac{dV^2}{dr} = -2\frac{GM}{r^2} \left(1 - \frac{c_s^2 r}{GM}\right),$$

where \( V \) is the radial velocity, and \( c_s \) the sound speed, which is usually a function of radius. Since this equation fully describes the outflow, it is called a wind equation. It
is this wind equation that we can integrate, with possible several supplementary equations, to obtain a full description of the wind, making it the most important equation in this thesis. The wind equation also divides the solutions into different families (see figure 2.1). This can be understood by looking at equation (2.5) a bit more closely. The overall form of the right hand side is that of a numerator and a denominator. When the flow velocity crosses the sound speed, the denominator goes through zero. If the numerator still has a finite value at that point, the acceleration of the flow becomes infinite. So in order to cross this point smoothly, the numerator has to be zero there as well. This constitutes an internal boundary condition, called a ‘regularity condition’, and therefore specifies the radius of this singular point, namely:

\[ r_S = \frac{GM}{2c_S^2}, \]  

(2.6)

where \( r_S \) is the spherical radius of this so-called sonic point. While it is usually called the sonic point because the wind equation is one-dimensional, because of the spherical symmetry, it is actually a spherical surface surrounding the star. It is convenient to express the radius \( r \) in units of the radius of the sonic point radius \( r_S \) and the velocity in units of the sound speed, that is to say, as the Mach number:

\[ M_S = \frac{V}{c_S}. \]  

(2.7)

There is also a physical way to approach this issue. Since the flow is hydrodynamic, there is a characteristic velocity, the sound speed. As the flow velocity exceeds the sound speed, the flow downstream can no longer affect the flow upstream. There is no signal that can travel fast enough to propagate upstream to the star. In other words the flow upstream is causally disconnected from the flow downstream of the sonic point and the surface it defines is called a separatrix surface, since it separates two modes of behaviour, the subsonic and the supersonic regime. This is a general feature of the wind equation: where there are mathematical singularities, there are corresponding physical separatrix surfaces.

The velocity in the Parker wind equation is squared everywhere, so the wind equation describes an outflow, or wind, as well as an inflow, or accretion. There are thus two solutions crossing the sonic point, labelled I for the wind and II for the accretion mode. These two lines divide the diagram in four regions. Regions III and IV are double-valued, a result of the numerator not being zero, where the denominator is, causing infinite acceleration and turning the flow back on itself. This behaviour is unphysical and thus it cannot be a proper solution. Regions V and VI have flow velocities that are supersonic or subsonic throughout the domain, with a minimum and maximum velocity respectively. These extrema are realised when the
Figure 2.1: Families of solutions of the Parker wind equation. The velocity in units of the sound speed is plotted against the radius in units of the sonic point radius. The thick lines labelled I and II are the wind and accretion solution respectively, dividing the plot into four separate regions. Every region, denoted by III – VI, has one typical solution plotted for that region, but is actually completely filled with solutions. See the text for details on the different solution families.

The numerator is zero, while the denominator is still finite, since the flow velocity is never equal to the sound speed. While region V is unrealistic with supersonic velocities far away from the central object, solutions in region VI could be possible and are called breeze solutions. Although all regions can be part of the solutions in shock transitions due to their discontinuous nature, in this thesis we will focus on winds with smooth transitions of the singular point(s), called type I solutions in figure 2.1.

2.1.2 Non-relativistic cold cylindrically symmetric MHD wind

The first non-relativistic magnetohydrodynamic wind model was derived by Blandford & Payne (1982). As the aim was to describe a jet, a cylindrical geometry ($\sigma, \phi, z$) was adopted. The matter is cold and the acceleration is caused by the centrifugal force as it is scooped up by the field lines. Since the general MHD equations are too complicated to solve for the general case, several assumptions were made in order to reduce the number of free parameters to be solved for. These are time independence; ideal MHD, giving

$$ E + V \times B = 0, $$

(2.8)
2.1 History

Figure 2.2: Example of self-similarity. All poloidal field lines (solid) cross radial lines (dotted) at the same angle. Also the Alfvén point, indicated by the dashed line labelled $M = 1$, lies on a radial line with a specific angle $\theta$.

corresponding to a high magnetic Reynolds number, meaning the matter strictly follows the magnetic field lines; axisymmetry around the $z$-axis; and self-similarity, meaning all field lines have the same shape and can be obtained by scaling one reference field line, although the magnetic field strength and density have separate scalings (see figure 2.2). For a flow that is steady and axisymmetric, the poloidal velocity and magnetic field are parallel to each other

$$V_p = \frac{\Psi_A(A)}{4\pi \rho} B_p,$$

(2.9)

where $\Psi_A$ is the mass-to-magnetic flux function, $A$ is the magnetic flux function, and $B_p$ is the poloidal magnetic field. The magnetic flux function can be thought of as a circular surface centred on the $z$-axis, with a value corresponding to the number of magnetic field lines crossing this surface, being zero for zero radius. This value changes depending on the radius of the surface and its height, but due to the axisymmetry and since field lines do not cross, the value is the same along a particular field line, and thus a field line can be labelled by the value of this function. The toroidal velocity is provided by the angular velocity of the field line ($\Omega$), which is also constant.
along a particular field line, leading to an expression for the full velocity

\[ \mathbf{V} = \frac{\Psi_A}{4\pi\rho} \mathbf{B} + \varpi \Omega \hat{\phi}. \] (2.10)

There are two other constants of motion for a particular field line, namely the specific energy:

\[ E = \frac{V^2}{2} + h + \Phi - \frac{\varpi B_\phi}{\Psi_A}, \] (2.11)

where \( E \) is the energy density, \( h \) is the enthalpy per unit mass, and \( \Phi \) is the gravitational potential, and the specific angular momentum:

\[ L = \varpi V_\phi - \frac{\varpi B_\phi}{\Psi_A}. \] (2.12)

Using these expressions it is possible to define three dimensionless parameters that are constant along a field line, describing the energy density:

\[ \epsilon_{BP} \equiv \frac{E}{(GM/\varpi_i)^{1/2}}, \] (2.13)

the angular momentum:

\[ \lambda_{BP} \equiv \frac{L}{(GM\varpi_i)^{1/2}}, \] (2.14)

and the mass-to-magnetic flux ratio:

\[ \kappa_{BP} \equiv \Psi_A \left[ 1 + \frac{1}{\tan^2(\psi_i)} \right]^{1/2} \frac{(GM/\varpi_i)^{1/2}}{B_i}, \] (2.15)

where \( \psi \) is the angle the field line makes with the disc, and a subscript \( i \) indicates a quantity evaluated at the disc surface. When the flow is adiabatic with polytropic index \( \Gamma \), a fourth dimensionless constant can be defined:

\[ \mu_{BP} \equiv \frac{P}{(B_i^2/4\pi)} \left( \frac{B_i^2 \varpi_i}{4\pi \rho GM} \right)^\Gamma, \] (2.16)

describing the thermal energy density. Since the flow is cold, this constant is equal to zero in the article.

Since we are now dealing with magnetic fields, a new characteristic velocity enters the problem. This Alfvén velocity, in the poloidal direction given by:

\[ V_{A,p} = \frac{B_p}{\sqrt{4\pi\rho}} \] (2.17)
plays a similar role for magnetic fields as the sound speed does for gas, and analogously we define a poloidal Alfvénic Mach number:

\[ M = \sqrt{\frac{4\pi \rho}{B_p}}. \] (2.18)

There is another velocity present in MHD flow, the fast magnetosonic velocity, with a Mach number given by:

\[ M_f = \sqrt{\frac{4\pi \rho}{B}}. \] (2.19)

Again using the continuity equation (2.1) and the z-component of the momentum equation (2.2), it is possible to derive a wind equation. Thanks to the presence of magnetic fields, this wind equation has many more terms, but the overall form, a single numerator and denominator, is preserved. The denominator in this case consists of the product of two subtractions, \((M^2 - 1)(M^2_f - 1)\), where \(M_{f,\theta}\) is the fast magnetosonic Mach number in the \(\theta\)-direction, towards the \(z\)-axis. This product means the denominator can go to zero at two points, where the flow velocity reaches the Alfvén velocity, called the Alfvén point (AP), and where the flow velocity in the \(\theta\)-direction attains the fast magnetosonic velocity, called the modified fast point (MFP, so called because the name fast magnetosonic point is reserved for the location where the poloidal velocity reaches the fast magnetosonic velocity). While they are called points because they occur on a one-dimensional field line, due to the self-similarity and axisymmetry assumption they are actually surfaces, more specifically cones around the \(z\)-axis. The reason for these specific conditions lie in the same assumptions. Because according to self-similarity all flow properties must be the same along a radial line and due to axisymmetry they are the same along a toroidal line, the only direction a wave can propagate is the \(\theta\)-direction. Since a separatrix surface forms at the location where the flow velocity matches the wave velocity in the direction of propagation, it is the flow velocity in the \(\theta\)-direction, not the poloidal direction, that has to be compared to the different characteristic velocities. As Alfvén waves are purely magnetic and \(V_p || B_p\), they can travel in any direction in the meridional plane and therefore also \(V_p = V_{A,p}\) at the AP.

Beyond the AP Alfvén waves cannot propagate backwards towards the central object, so all the APs indeed form a separatrix surface. However the fast magnetosonic speed exceeds the Alfvén velocity, so it is still possible to transmit information from beyond the Alfvén point to the central object and the flow upstream of the AP is not necessarily causally disconnected from the flow downstream.

Since the fast magnetosonic velocity is the fastest velocity at which any signal can travel, the flow does become causally disconnected at the fast magnetosonic separatrix surface (FMSS), formed by all the MFPs, where \(M^2_{f,\theta} = 1\).
2 Background and Methodology

Blandford & Payne (1982) were interested in solutions where the flow became collimated parallel to the z-axis. This constraint meant that the flow velocity in the \(\theta\)-direction never became very large and consequently that \(M_{l,\theta}^2 < 1\) throughout the flow. Therefore the second singular point was not crossed. The flow was also cold, which means that the third singular point of MHD flows was not treated by these authors.

2.1.3 Relativistic cold cylindrically symmetric MHD wind

Ten years later Li et al. (1992) constructed a class of self-similar solutions for relativistic winds. The relativistic treatment introduces a natural length scale, the light cylinder radius. If field lines were to rotate as a spoked wheel, dragging the matter with them, the matter would rotate with the velocity of light at a certain radius given by:

\[
\varpi_L = \frac{c}{\Omega_i},
\]

where \(c\) is the velocity of light, and \(\Omega_i\) is the angular velocity of the part of the disc where the field line is anchored. Since this is physically impossible, the field lines have to bend backwards with respect to the rotation, increasing the azimuthal magnetic field.

By filling in the identity of the Lorentz factor, it is possible to obtain an equation that describes the partition of energy in the system. This can be referred to as the energy equation, but is also sometimes called the Bernoulli equation. If the geometry is known, this equation immediately gives the acceleration.

Another relativistic effect is the importance of an electric field. The direction of the electric field is in the poloidal direction and perpendicular to the poloidal velocity streamlines, called the transfield direction, and therefore the electric field does not cause any additional acceleration. It does however affect the collimation of the flow. By projecting the relativistic momentum equation onto a unit vector in the transfield direction it is possible to write down the full transfield force balance equation, which determines the geometry of the field lines. Therefore together with the energy equation this equation completely describes the system and from them it is possible to derive the wind equation.

In this article solutions were sufficiently far away from the central object for gravity and gas pressure to be neglected and it is actually these simplifications that allow self-similar solutions to be found in the relativistic case. The downside is that the region near the compact object is not very well described. Another side effect of the self-similarity assumption is that the angular velocity of the disc falls of as \(\varpi^{-1}\), which is more restrictive than the non-relativistic case, as Keplerian accretion discs can no longer be modelled.
2.1.4 Non-relativistic warm cylindrically symmetric MHD wind

Another eight years later the cold Blandford & Payne model was generalised by Vlahakis et al. (2000) to allow the flow to be hot. They assumed a polytropic relationship between the gas pressure and the density

\[ Q = \frac{P}{\rho \Gamma}, \]  

(2.21)

where \( Q \) is the specific entropy, and a parameter proportional to the gas entropy, describing the ratio of the gas pressure to the magnetic energy density at the Alfvén point:

\[ \mu_{\text{VTST}} = \frac{8\pi P_A}{B_A^2}. \]  

(2.22)

Both \( Q \) and \( \mu_{\text{VTST}} \) are constant along a field line.

While a similar parameter was present in Blandford & Payne (1982), in this article it is used for the first time in a calculation. The effect of a warm flow is that a third singular point appears in the wind equation. When the denominator of the wind equation becomes zero, the velocity \( V_\theta \) satisfies the quartic:

\[ V_{\theta}^4 - V_{\theta}^2 \left( c_S^2 + V_A^2 \right) + c_S^2 V_{A,\theta}^2 = 0, \]  

(2.23)

in other words, where \( V_{\theta}^2 \) equals:

\[ V_{s}^2 = \frac{1}{2} \left\{ c_S^2 + V_A^2 - \left[ \left( c_S^2 + V_A^2 \right)^2 - 4 c_S^2 V_{A,\theta}^2 \right]^{1/2} \right\}, \]  

(2.24)

called the slow magnetosonic velocity, or:

\[ V_{f}^2 = \frac{1}{2} \left\{ c_S^2 + V_A^2 + \left[ \left( c_S^2 + V_A^2 \right)^2 - 4 c_S^2 V_{A,\theta}^2 \right]^{1/2} \right\}, \]  

(2.25)

called the fast magnetosonic velocity. The first equality corresponds to the slow magnetosonic separatrix surface (SMSS), or the modified slow point (MSP). In a cold flow the sound speed is zero, and consequently the slow magnetosonic velocity is 0 as well. For a cold flow the MSP thus lies at zero height, which is why Blandford & Payne (1982) were unable to cross it.

By starting the integration at the Alfvén point and integrating inward towards the MSP and outward towards the MFP, it was possible to cross all three singular points in the flow, leading to a credible jet solution spanning from below the MSP to beyond the MFP. All that remained was to allow the flow to become relativistic.
2.1.5 Relativistic warm cylindrically symmetric MHD wind

The equations for a relativistic hot flow were written down by Vlahakis & Königl (2003a). The assumptions remained the same, time-independence, ideal MHD, axisymmetry, and self-similarity, with the addition of a zero azimuthal electric field ($E_\phi = 0$), which is generated due to relativistic effects. The velocity is given by:

$$V = \frac{V_A}{4\pi\gamma\rho_0} B + \sigma \Omega \phi, \quad \frac{V_p}{B_p} = \frac{V_A}{4\pi\gamma\rho_0},$$

(2.26)

where $\rho_0$ is the baryon rest-mass density. The five parameters constant along a field line are the field angular velocity:

$$\Omega = \frac{V_\phi}{\sigma} - \frac{V_A}{4\pi\gamma\rho_0} \frac{B_\phi}{\sigma},$$

(2.27)

the mass-to-magnetic flux ratio:

$$\Psi_A = \frac{4\pi\gamma\rho_0 V_\phi}{B_p},$$

(2.28)

the total (kinetic plus magnetic) specific angular momentum:

$$L = \xi \gamma \sigma V_\phi - \frac{\sigma B_\phi}{\Psi_A},$$

(2.29)

where $\xi c^2$ is the specific (per baryon mass) relativistic enthalpy, the total energy-to-mass flux ratio $\mu c^2$ with:

$$\mu = \xi \gamma - \frac{\sigma \Omega B_\phi}{\Psi_A c^2},$$

(2.30)

and the specific entropy:

$$Q = \frac{P}{\rho_0^\Gamma}.$$

(2.31)

Since gravity had again to be ignored to allow self-similar relativistic flow, the modified slow point was not solved for. For the far-field solution it was deemed an asymptotically cylindrical flow was the only physically acceptable solution, with the MFP at infinite height. So in effect only the Alfvén point was crossed.

The free parameters in this model are $F$, which controls the current distribution and is given by:

$$F = 1 + \frac{\text{d} \log(I)}{\text{d} \log(r)},$$

(2.32)

where $I$ is the current, $\Gamma$, the adiabatic index, $\theta_A$, the poloidal spherical angle of the Alfvén point, $\psi_A$, the angle of the field line at the Alfvén point, $x_A$, the cylindrical
2.2 Extending the model

We would like to derive a model that allows us to cross all three critical points in a relativistic flow, so it is possible to relate the height of the MFP to properties close to the black hole (around the MSP). To this end we want to compare the relativistic wind equation of Vlahakis & Königl (2003a, hereafter VK) without gravity, with the wind equation given in Vlahakis et al. (2000, hereafter VTST) which does include gravity, in order to obtain a gravity term, which includes only the kinetic inertia, and neglects the thermal and electromagnetic inertias (Polko et al. 2013a). We will call this term the kinetic gravity term. We need to derive the VK wind equation, since this equation is not given within the paper. We also want to derive a gravity term that includes all inertias, called the full gravity term, and compare it with the kinetic gravity term. For this we need to rederive new forms of the relativistic energy and transfield equations. We will also derive the non-relativistic transfield equation for completeness.

<table>
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<th>Gravity</th>
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Table 2.1: Overview of radially self-similar wind models, indicating whether the flow has thermal energy and whether the flow can attain relativistic velocities. Also indicated is which singular points are crossed, named the slow magnetosonic separatrix surface (SMSS), or modified slow point (MSP), the Alfvén surface (AS), and the fast magnetosonic separatrix surface (FMSS), or modified fast point (MFP).
2 Background and Methodology

2.2.1 Non-relativistic wind model with gravity

One way to obtain the wind equation from the energy and transfield equation is by using the determinant method. When we write both these equations in the form:

\[ A \frac{dM^2}{d\theta} + B \frac{d\psi}{d\theta} = C, \quad (2.33) \]

using subscripts 1 for the energy equation and subscripts 2 for the transfield equation, we have two equations and two unknowns, leading to the solutions:

\[ \frac{dM^2}{d\theta} = \frac{C_1B_2 - C_2B_1}{A_1B_2 - A_2B_1} \]

and:

\[ \frac{d\psi}{d\theta} = \frac{C_2A_1 - C_1A_2}{A_1B_2 - A_2B_1}. \]

The first equation is the wind equation. Since every product in both the numerator and the denominator of the formula is composed of a single term from the energy equation and a single term from the transfield equation, if we multiply all terms from one equation with the same factor, this factor will cancel out in the final equation. We will make use of this to simplify the calculations. To show this method works, we test it on the VTST equations, since that wind equation is already given. We write down the terms obtained from the derivative of the energy equation, derive the transfield equation and find its terms, and then calculate the wind equation.

Terms from the energy equation

Here we give the expressions for the determinant method obtained from the derivative of the energy equation with respect to \( \theta \):

\[ A_1 = \left[ \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \right] \times \left[ -2\lambda^2_{\text{VTST}} \frac{M^2(1 - G^2)^2 \cos^2(\psi + \theta)}{G^2(1 - M^2)^3 \sin^2(\theta)} \right. \]
\[ \left. + \frac{\mu_{\text{VTST}} \cos^2(\psi + \theta)}{M^{2F} \sin^2(\theta)} - 2 \frac{M^2}{G^4} \right]. \]

\[ B_1 = \left[ \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \right] \left[ -2 \frac{M^4}{G^4} \tan(\psi + \theta) \right], \]

\[ C_1 = \left[ \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \right] \left[ \frac{2\kappa_{\text{VTST}}^2 \sin(\psi + \theta)}{G} \right. \left. \frac{\sin(\psi + \theta) \cos(\psi + \theta)}{\sin^2(\theta)} \right. \]
\[ - 2 \left. \frac{M^4}{G^4} \frac{\cos(\psi)}{\sin(\psi + \theta) \cos(\psi + \theta)} \right] + 2\lambda^2_{\text{VTST}} \frac{(2M^2 - 1)G^4 - M^4 \cos(\psi) \cos(\psi + \theta)}{G^2(1 - M^2)^2 \sin^3(\theta)}. \]

The common factors have been extracted to make the eventual multiplication and comparison easier.
2.2 Extending the model

Derivation of the transfield equation

The transfield equation is the inner product of the sum of forces acting on a field line and the unit vector normal to the field line. The forces are the kinetic force, the thermal pressure force, the electromagnetic force and the gravitational force:

\[-\rho(V \cdot \nabla)V - \nabla P + \frac{(\nabla \times B) \times B}{4\pi} + \rho \nabla \frac{G M}{r} = 0.\] \hspace{1cm} (2.39)

Taking the inner product, the kinetic force becomes:

\[-\rho(V \cdot \nabla)V \cdot \hat{n} = \left[-M^2 \sin^2(\theta) \frac{\partial \psi}{\partial \theta} - \lambda_{\text{VST}}^2 \frac{G^2 (G^2 - M^2)^2}{M^2 (1 - M^2)^2} \sin(\psi + \theta) \sin(\theta)\right] \times \frac{B_0^2 \alpha^{F-2}}{4\pi \sigma G^4 \cos(\psi + \theta)}.\] \hspace{1cm} (2.40)

The thermal pressure force becomes:

\[-\nabla P \cdot \hat{n} = -\frac{\mu_{\text{VST}}}{M^2} G^4 \left[(F - 2) + \sin(\psi + \theta) \cos(\psi + \theta) \frac{F}{2M^2} \frac{\partial M^2}{\partial \theta}\right] \times \frac{B_0^2 \alpha^{F-2}}{4\pi \sigma G^4 \cos(\psi + \theta)}.\] \hspace{1cm} (2.41)

The electromagnetic force becomes:

\[
\frac{(\nabla \times B) \times B}{4\pi} \cdot \hat{n} = \left\{ \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \left[ -(F - 2) - \frac{\cos(\psi) \sin(\psi + \theta)}{\sin(\theta)} + \frac{\partial \psi}{\partial \theta} \right]
+ \lambda_{\text{VST}}^2 \frac{G^2 (1 - G^2)^2}{(1 - M^2)^2} \left[ -(F - 1) - \frac{2G^2}{1 - G^2} \frac{\cos(\psi) \sin(\psi + \theta)}{\sin(\theta)} \right]
+ \frac{\sin(\psi + \theta) \cos(\psi + \theta) \partial M^2}{1 - M^2} \frac{\partial \theta}{\partial \theta}\right\} \frac{B_0^2 \alpha^{F-2}}{4\pi \sigma G^4 \cos(\psi + \theta)}.\] \hspace{1cm} (2.42)

And the gravitational force becomes:

\[
\rho \nabla \frac{G M}{r} \cdot \hat{n} = -\frac{\kappa_{\text{VST}}^2 \sin(\theta) G^4}{G} \frac{G^2 \cos^2(\psi + \theta)}{M^2} \frac{B_0^2 \alpha^{F-2}}{4\pi \sigma G^4 \cos(\psi + \theta)}.\] \hspace{1cm} (2.43)
2 Background and Methodology

The full transfield equation is then given by:

\[
\begin{aligned}
&\left\{ -M^2 \sin^2(\theta) \frac{\partial \psi}{\partial \theta} - \lambda_{VTST}^2 \frac{G^2(G^2 - M^2)^2}{M^2(1 - M^2)^2} \sin(\psi) \frac{\cos(\psi + \theta)}{\sin(\theta)} \\
&\quad - \frac{\mu_{VTST}}{M^2 G^4} \left[ (F - 2) + \sin(\psi + \theta) \cos(\psi + \theta) \frac{\Gamma}{2M^2} \frac{\partial M^2}{\partial \theta} \right] \\
&\quad + \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \left[ -(F - 2) - \frac{\cos(\psi) \sin(\psi + \theta)}{\sin(\theta)} + \frac{\partial \psi}{\partial \theta} \right] \\
&\quad + \lambda_{VTST}^2 \frac{G^2(1 - G^2)^2}{(1 - M^2)^2} \\
&\quad \times \left[ -(F - 1) - \frac{2G^2}{1 - G^2} \sin(\psi) \sin(\psi + \theta) + \frac{\sin(\psi + \theta) \cos(\psi + \theta) \partial M^2}{1 - M^2} \right] \\
&\quad - \frac{\kappa_{VTST}^2}{G} \frac{\sin(\theta)}{M^2} \cos^2(\psi + \theta) \right\} \frac{B^2 \phi^{F-2}}{4\pi \sigma G^4} \frac{\sin(\theta)}{\cos(\psi + \theta)} = 0. \tag{2.44}
\end{aligned}
\]

And the individual components used for the determinant method:

\[
A_2 = \left[ \frac{B^2 \phi^{F-2}}{4\pi \sigma G^4} \frac{\sin(\theta)}{\cos(\psi + \theta)} \right] \lambda_{VTST}^2 \frac{G^2(1 - G^2)^2}{(1 - M^2)^2} \sin(\psi + \theta) \cos(\psi + \theta) \\
\quad - \Gamma \mu_{VTST} M^{-2G} \frac{G^4}{2M^2} \sin(\psi + \theta) \cos(\psi + \theta), \tag{2.45}
\]

\[
B_2 = \left[ \frac{B^2 \phi^{F-2}}{4\pi \sigma G^4} \frac{\sin(\theta)}{\cos(\psi + \theta)} \right] \sin^2(\theta) \frac{1}{\cos^2(\psi + \theta)} - M^2, \tag{2.46}
\]

\[
C_2 = \left[ \frac{B^2 \phi^{F-2}}{4\pi \sigma G^4} \frac{\sin(\theta)}{\cos(\psi + \theta)} \right] \lambda_{VTST}^2 \frac{G^2(G^2 - M^2)^2}{M^2(1 - M^2)^2} \sin(\psi) \cos(\psi + \theta) \\
+ \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \left[ (F - 2) + \frac{\cos(\psi) \sin(\psi + \theta)}{\sin(\theta)} \right] \\
+ \lambda_{VTST}^2 \frac{G^2(1 - G^2)^2}{(1 - M^2)^2} \left[ (F - 1) + \frac{2G^2}{1 - G^2} \cos(\psi) \sin(\psi + \theta) \right] \\
+ \frac{\mu_{VTST}}{M^2 G^4} (F - 2) G^4 + \frac{\kappa_{VTST}^2 \sin(\theta)}{G} \frac{G^4}{M^2} \cos^2(\psi + \theta). \tag{2.47}
\]
The wind equation is then given by \((C_1 B_2 - C_2 B_1) / (A_1 B_2 - A_2 B_1)\):  

\[
\frac{dM^2}{d\theta} = -2 \tan(\psi + \theta) \left[ -\frac{\kappa_{VTST}^2 \sin(\theta)}{G} - \mu_{VTST} (F - 2) M^{4-2\Gamma} \right.
\]

\[
+ \frac{M^4}{G^4} (1 - M^2) \frac{\cos(\psi) \sin(\theta)}{\sin(\psi + \theta)} - \frac{M^4}{G^4} (F - 2) \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)}
\]

\[
- \lambda_{VTST}^2 \frac{M^4}{G^2} (F - 2) \left( \frac{1 - G^2}{1 - M^2} \right)^2
\]

\[
+ \lambda_{VTST}^2 \frac{M^2 G^4 - M^4}{G^2 (1 - M^2)}
\]

\[
- \lambda_{VTST}^2 \frac{\cos(\psi)}{\sin(\theta) \sin(\psi + \theta)} \frac{(2M^2 - 1)G^4 - M^4}{G^2 (1 - M^2)}
\]

\[
\times \left\{ \left[ \frac{\Gamma \mu_{VTST}}{2} \frac{1 - M^2}{M} - \lambda_{VTST}^2 \frac{M^2 G^2}{G^4} \left( \frac{1}{1 - M^2} \right) \right]^2
\]

\[
+ \frac{M^4 \sin^2(\theta)}{G^4} - \frac{M^2 \sin^2(\theta)}{G^4 \cos^2(\psi + \theta)} \right\}^{-1},
\]

(2.48)

which is, indeed, the same as given in VTST.

### 2.2.2 Relativistic wind model without gravity

The right-hand side of the wind equation consists of a numerator and a denominator. Only the denominator is given in VK, so we will first derive the denominator using the determinant method and compare it with the one given, showing this method works. After a successful comparison, we will derive the numerator to obtain the full wind equation. As above, we will label the terms from the energy equation with the subscript 1 and those from the transfield equation with subscript 2.
Terms from the energy equation

The terms from the derivative of the energy equation with respect to $\theta$, with the common factor $-2F^2\sigma_M^2 G^6(1 - M^2 - x^2)^2 \sin^2(\theta) \tan(\psi + \theta)$, are:

$$A_1 = \frac{\mu^2 \chi^2}{F^2 \sigma_M^2} M^2 \frac{(1 - G^2)^2}{G^2 (1 - M^2 - x^2)^3} \frac{\cos^3(\psi + \theta)}{\sin(\psi + \theta)} + \frac{M^2 \cos(\psi + \theta)}{G^2 \sin(\psi + \theta)}$$

$$- \frac{x^2}{F^2 \sigma_M^2} \frac{\cos^3(\psi + \theta)}{(2 - \Gamma) (\xi - 1) \sin(\psi + \theta)}.$$  \hspace{1cm} (2.49)

$$B_1 = \frac{M^4}{G^4}.$$  \hspace{1cm} (2.50)

$$C_1 = \frac{\xi^2 \chi^4}{F^2 \sigma_M^2} \frac{\cos(\psi) \cos^2(\psi + \theta)}{\sin^3(\theta) \sin(\psi + \theta)} \left\{ \frac{\mu^2 G^4(1 - M^2 - x^2)^2}{\xi^2 G^4(1 - M^2 - x^2)^2} - \frac{x^2}{1 - M^2 - x^2} \frac{\cos^2(\psi + \theta) - 2 M^2}{G^4(1 - M^2 - x^2)^2} \right\}$$

$$- \frac{x^2}{\xi^2} (\frac{1 - M^2 - x^2)^2}{G^4(1 - M^2 - x^2)^2} (1 - M^2 - x^2) (1 - x^2).$$  \hspace{1cm} (2.51)

Terms from the transfield equation

The terms from the transfield equation are:

$$A_2 = \left[ \frac{B_0^2 \alpha F^2}{4 \pi \sigma G^4} \right] \frac{\sin(\theta)}{\cos(\psi + \theta)} \frac{\xi^2 \chi^4}{F^2 \sigma_M^2} \sin(\psi + \theta) \cos(\psi + \theta)$$

$$\times \left[ \frac{\mu^2 x^2 (1 - G^2)^2}{\xi^2 (1 - M^2 - x^2)^3} - \frac{(\Gamma - 1) (\xi - 1)}{\xi - (\Gamma - 1) (\xi - 1) M^4} \right].$$  \hspace{1cm} (2.52)

$$B_2 = \left[ \frac{B_0^2 \alpha F^2}{4 \pi \sigma G^4} \right] \frac{\sin(\theta)}{\cos(\psi + \theta)} \left[ \frac{1}{\cos^2(\psi + \theta) - M^2} \right].$$  \hspace{1cm} (2.53)

$$C_2 = \left[ \frac{B_0^2 \alpha F^2}{4 \pi \sigma G^4} \right] \frac{\sin(\theta)}{\cos(\psi + \theta)} \left[ \frac{x^2 \mu^2 x^2}{(F - 2 - F x^2 + x^2 + (1 + x^2) \cos(\psi) \sin(\psi + \theta)) \sin(\theta)} \right]$$

$$+ \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \left[ x^2 \mu^2 x^2 \left( \frac{1 - G^2}{1 - M^2 - x^2} \right) \left( F - 1 \right) + \frac{2 G^2}{1 - G^2} \left( \frac{1 - M^2 - x^2}{1 - M^2 - x^2} \right) \cos(\psi) \sin(\psi + \theta) \right]$$

$$+ \frac{2 \Gamma - 1}{\Gamma} \left[ \frac{1 - G^2}{F^2 \sigma_M^2} \left( \frac{1 - G^2}{1 - M^2 - x^2} \right) \right].$$  \hspace{1cm} (2.54)
2.2 Extending the model

The denominator of the wind equation

The denominator from the determinant method is:

\[ D_{\text{DM}} = \left[ 2F^2\sigma_M^2G^6(1 - M^2 - x^2)^2\sin^2(\theta) \right] \times \left[ \frac{x_A^4}{F^2\sigma_M^2(2 - \Gamma)} \xi^2(\xi - 1) \frac{(1 - M^2 - x^2)}{M^2} - \frac{\mu x_A^6 M^2}{F^2\sigma_M^2 G^2 (1 - M^2 - x^2)^2} + \frac{M^4\sin^2(\theta)}{G^4} \right] \]

\( (2.55) \)

We want to compare the denominator we obtain from applying the determinant method (with subscript DM) to the denominator given by equation (26) in VK (with subscript VK):

\[ D_{\text{VK}} = \left[ \frac{x_A^4}{F^2\sigma_M^2(2 - \Gamma)} \xi^2(\xi - 1) \frac{(1 - M^2 - x^2)}{M^2} - \frac{\mu x_A^6 M^2}{F^2\sigma_M^2 G^2 (1 - M^2 - x^2)^2} + \frac{M^4\sin^2(\theta)}{G^4} \right] \frac{\gamma}{\gamma^2 G^4 x_A}. \]

\( (2.56) \)

Apart from the overall scaling, which means the numerators would also have a different scaling, the two denominators are the same. By comparing with the denominator of the VTST wind equation in (2.48), we can see that we have extracted the right scaling.

The numerator of the wind equation

Based on the successful comparison of the denominators, we will now try to match the numerators as well as possible, by writing both the VTST and VK wind equations in the form that makes them most similar in structure. In order to find the correct normalisation (as terms can be arbitrarily put in the numerator or denominator), we look at the extracted factors to match the denominators. In formulas:

\[ \frac{A_{\text{DM}}}{D_{\text{DM}}} = \frac{C_1 B_2 - C_2 B_1}{A_1 B_2 - B_1 A_2} = \frac{A \cdot \text{CN}}{B \cdot \text{CD}} = \frac{C \cdot \text{CN}}{D \cdot \text{CD}} = \frac{A_{\text{VTST}}}{D_{\text{VTST}}} : \text{VTST}, \]

where DM stands for determinant method and CN and CD stand for common numerator and common denominator respectively. A is what we want to determine, B is...
given by equation (2.55), and C and D by equation (2.48):

\[ B = 2F^2\sigma_M^2G^6(1 - M^2 - x^2)^2 \sin^2(\theta), \]  
\[ C = -2\frac{\sin(\psi + \theta)}{\cos(\psi + \theta)} = -2\tan(\psi + \theta), \]  
\[ D = 2, \]  
\[ A = \frac{CB}{D} = -2F^2\sigma_M^2G^6(1 - M^2 - x^2)^2 \sin^2(\theta) \tan(\psi + \theta). \]  

So, equation (2.61) is the factor that should be extracted from the numerator of the VK wind equation for it to be compared to the VTST numerator. Writing out the numerator of the VK wind equation \((C_1B_2 - C_2B_1)\):

\[ \mathcal{N}_{\text{DM}} = -2\frac{\Gamma - 1}{\Gamma} \frac{x_A^2}{F^2\sigma_M^2} \bar{\xi}(\bar{\xi} - 1)(F - 2)M^2 + \frac{M^4}{G^4}(1 - M^2 - x^2) \cos(\psi) \sin(\theta) \]
\[ - \frac{M^4}{G^4}(F - 2 - Fx^2 + x^2) \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} - 2x^2 \frac{M^4}{G^4} \cos(\psi) \sin(\theta) \sin(\psi + \theta) \cos^2(\psi + \theta) \]
\[ + \frac{\mu^2 x_A^2 M^2}{F^2\sigma_M^2 G^2} \left[ \frac{G^2 - M^2 - x^2}{1 - M^2 - x^2} \right] \]
\[ - \frac{\mu^2 x_A^2}{F^2\sigma_M^2 G^2} \cos(\psi) \left[ \frac{G^2 - M^2 - x^2 + 2G^2 M^2(1 - G^2)}{G^2(1 - M^2 - x^2)} \right]. \]

Now we identify all parts separately with the VTST parts, taking the non-relativistic \((V \ll c)\), and the non-force-free MHD \((x = 0)\) limits:

\[ -2\frac{\Gamma - 1}{\Gamma} \frac{x_A^2}{F^2\sigma_M^2} \bar{\xi}(\bar{\xi} - 1)(F - 2)M^2 = -\mu_{\text{VTST}}(F - 2)M^{4-2\Gamma}, \]
\[ \frac{M^4}{G^4}(1 - M^2 - x^2) \frac{0}{\cos(\psi) \sin(\theta)} \sin(\psi + \theta) = \frac{M^4}{G^4}(1 - M^2) \cos(\psi) \sin(\theta) \sin(\psi + \theta), \]
\[ - \frac{M^4}{G^4} \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \left[ F - 2 - Fx^2(F - 1) \right] = -\frac{M^4}{G^4}(F - 2) \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)}, \]
\[ \frac{\mu^2 x_A^2 M^2}{F^2\sigma_M^2 G^2} \left[ \frac{(G^2 - M^2 - x^2)^0}{1 - M^2 - x^2} \right] \]
\[ = -\frac{\mu^2 x_A^2 M^4}{F^2\sigma_M^2 G^2} \frac{(1 - G^2)^2}{1 - M^2} + \frac{\mu^2 x_A^2 M^4}{F^2\sigma_M^2 G^2} \frac{G^4 - M^2}{1 - M^2}. \]
2.2 Extending the model

\[
\left[ \frac{(G^2 - M^2 - \hat{\mathcal{G}}^2)^0 + 2G^2M^2(1 - G^2)}{G^2(1 - M^2 - \hat{\mathcal{G}}^2)^0} \right] = -\frac{(2M^2 - 1)G^4 - M^4}{G^2(1 - M^2)}. \quad (2.67)
\]

For comparison we give the full numerator of the VTST wind equation:

\[\mathcal{N}_{\text{VTST}} = -2 \sin(\psi + \theta) \left\{ -\frac{\kappa_{\text{VTST}}}{G^2} \sin(\theta) + \frac{\kappa_{\text{VTST}}}{G^2} \sin(\theta) \right\} - \frac{M^4}{G^4} (1 - M^2) \frac{\cos(\psi) \sin(\theta)}{\sin(\psi + \theta)} - \frac{M^4}{G^2} (F - 2) \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} + \frac{\mu^2 x_A^6}{F^2 \sigma_M^2} M^4 \frac{1 - G^2}{1 - M^2} + \frac{\mu^2 x_A^6}{F^2 \sigma_M^2} M^4 \frac{G^4 - M^2}{1 - M^2} \left( \frac{2M^2 - 1)G^4 - M^4}{G^2(1 - M^2)} \right) \]. \quad (2.68)

The only term not accounted for is the gravity term, \(-\kappa_{\text{VTST}}^2 \sin(\theta)/G\). In order to use this term in the VK framework, we need a prescription for \(\kappa_{\text{VTST}}\) in terms of the VK variables. By comparing the equations for the velocity, magnetic field, density and pressure, we can obtain expressions in VK notation for \(V, B, \lambda_{\text{VTST}}, \rho, P\) from the parts between the square brackets:

\[V_{\text{VTST}} = \left[ V, \alpha^{-1/4} \right] \frac{M^2 \sin(\theta)}{G^2 \cos(\psi + \theta)} \hat{b} + V, \alpha^{-1/4} \lambda_{\text{VTST}} \frac{G^2 - M^2}{G(1 - M^2)} \hat{\phi}, \quad (2.69a)\]

\[V_{\text{VK}} = \left[ \frac{F \sigma_M}{\gamma x_A^2} \right] \frac{M^2 \sin(\theta)}{G^2 \cos(\psi + \theta)} \hat{b} + \mu x_A^6 \frac{G^2 - M^2 - x^2}{\gamma x^2 \cos(\psi + \theta)} \hat{\phi}, \quad (2.69b)\]

\[B_{\text{VTST}} = -\left[ B_0 \right] \alpha \frac{\mu x_A^6}{G^2 \cos(\psi + \theta)} \hat{b} + B_0 \alpha \frac{\mu x_A^6}{G^2 \cos(\psi + \theta)} \left[ \lambda_{\text{VTST}} \frac{1 - G^2}{G(1 - M^2)} \hat{\phi}, \quad (2.70a)\right]

\[B_{\text{VK}} = -\left[ B_0 \right] \alpha \frac{\mu x_A^6}{G^2 \cos(\psi + \theta)} \hat{b} + B_0 \alpha \frac{\mu x_A^6}{F \sigma_M} \frac{1 - G^2}{G(1 - M^2 - x^2)} \hat{\phi}, \quad (2.70b)\]

\[\rho_{\text{VTST}} = \alpha^{-3/2} \rho_\phi \frac{1}{M^2}, \quad (2.71a)\]

\[\rho_0 = \left[ \frac{B_0^2}{4 \pi F \sigma_M^2} \right] \frac{\mu x_A^6}{1 - M^2} \hat{\phi}, \quad (2.71b)\]
2 Background and Methodology

<table>
<thead>
<tr>
<th>Non-rel.</th>
<th>Relativistic</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$F$</td>
<td>Parameter that controls the current distribution</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$\Gamma$</td>
<td>Polytropic index</td>
</tr>
<tr>
<td>$B_*$</td>
<td>$B_0$</td>
<td>Reference magnetic field strength</td>
</tr>
<tr>
<td>$V_*$</td>
<td>$\frac{\alpha^{1/4}F\sigma_M\gamma}{\gamma^2\xi_A}$</td>
<td>Alfvén velocity</td>
</tr>
<tr>
<td>$\sigma_*$</td>
<td>$\sigma_0$</td>
<td>Reference length</td>
</tr>
<tr>
<td>$\sigma_A$</td>
<td>$\sigma_A$</td>
<td>Alfvén lever arm</td>
</tr>
<tr>
<td>$\rho_*$</td>
<td>$\frac{B_0^2 x_A^4}{4\pi F^2\sigma_M^2} \xi \alpha^{-1/2}$</td>
<td>Density at the Alfvén radius along the reference field line</td>
</tr>
<tr>
<td>$P_*$</td>
<td>$\frac{B_0^2 \Gamma - 1}{4\pi} \frac{x_A^4}{F^2\sigma_M^2} \xi (\xi - 1) \frac{M^2}{M^2} \alpha^{-1/2}$</td>
<td>Pressure at the Alfvén radius along the reference field line</td>
</tr>
</tbody>
</table>

Table 2.2: Corresponding non-relativistic and relativistic terms and their meaning.

\[
P_{VTST} = \alpha^{F-2-\Gamma(F-3/2)} P_* \left( \frac{\rho}{P_*} \right)^{\Gamma} = \left[ \frac{P_*}{M^{2\Gamma}} \right] \alpha^{F-2}, \quad (2.72a)
\]

\[
P_{VK} = \left[ \frac{B_0^2 \Gamma - 1}{4\pi} \frac{x_A^4}{F^2\sigma_M^2} \xi (\xi - 1) \frac{M^2}{M^2} \right] \alpha^{F-2}. \quad (2.72b)
\]

Table 2.2 lists all VTST terms and their corresponding VK terms.

**Definition of the constants**

With these conversions we can now also list the four constants of the non-relativistic model and express them in their corresponding relativistic form. They are $\kappa_{VTST}$, the gravity, or mass loss parameter, which we can now fill in; $\lambda_{VTST}$, the specific angular momentum of the flow in units of $V_* \sigma_*$, from equation (2.70b); $\mu_{VTST}$, the gas entropy parameter, by comparing the denominators in equations (2.48) and (2.55); and
2.2 Extending the model

\( \epsilon_{VTST} \), the Bernoulli constant, from the energy equation:

\[
\begin{align*}
\kappa_{VTST} &= \sqrt{\frac{G M}{\sigma A V_z^2}} = \sqrt{\frac{G M}{c^2 \sigma_A F^2 \sigma_M^2} \frac{(1 - M^2 - \chi_A^2)^2}{(1 - M^2 - \chi^2)^2}}, \quad (2.73) \\
\lambda_{VTST} &= \frac{\mu^2 x_A^6}{F^2 \sigma_M^2}, \quad (2.74) \\
\mu_{VTST} &= 2 \Gamma - 1 \frac{x_A^4}{F^2 \sigma_M^2} \frac{\xi^2 (\xi - 1)}{\xi - (\Gamma - 1)(\xi - 1)} \frac{M^{2T}}{M^2}, \quad (2.75) \\
\epsilon_{VTST} &= \left[ \frac{1}{2} \frac{\mu^2 x_A^6}{G^2 \sigma_M^2} \frac{(1 - \chi_A^2)}{\chi - (\Gamma - 1)(\chi - 1)} - \frac{\mu^2 x_A^6}{G^2 \sigma_M^2} \frac{1 - G^2}{1 - M^2} \right]. \quad (2.77)
\end{align*}
\]

2.2.3 The Alfvén Regularity Condition

Since many terms in the numerator and denominator of the wind equation are of the form \( 0/0 \) at the Alfvén point, we need to rewrite the wind equation to start off the integration. First we define the parameter \( \rho_A \) to have the value of \( dM^2/d\theta \) at the Alfvén point. The procedure is to fill in the Alfvén values for all singular terms in the wind equation. The resulting equation, which is called the Alfvén Regularity Condition (ARC), can be solved for \( \rho_A \), although this generally has to be done numerically. We will derive the ARC in VK. We will introduce a place holder gravity term \( (G) \) in the numerator of the wind equation to show how it gets manipulated.

43
2 Background and Methodology

Derivation of the relativistic Alfvén Regularity Condition

The full VK wind equation is given by:

\[
\frac{dM^2}{d\theta} = -2\frac{\Gamma - 1}{\Gamma} (F - 2) \frac{x_A^4}{F^2\sigma_M^2} \xi (\xi - 1) M^2 + \frac{M^4}{G^4} \left( 1 - M^2 - x^2 \right) \cos(\psi) \sin(\theta) \sin(\psi + \theta) \\
- \frac{M^4}{G^4} \left( F - 2 - Fx^2 + x^2 \right) \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} - 2x^2 \frac{M^4 \cos(\psi) \sin(\theta) \sin(\psi + \theta)}{G^4} \cos(\psi + \theta) \\
+ \frac{\mu^2 x_A^4}{F^2\sigma_M^2} M^2 \left[ \left( G^2 - M^2 - x^2 \right)^2 - (F - 1) M^2 \left( 1 - G^2 \right)^2 \right] \\
+ \frac{\mu^2 x_A^4}{F^2\sigma_M^2} \cos(\psi) \frac{\sin(\theta)}{\sin(\psi + \theta)} \left[ \frac{G^2 - M^2 - x^2}{G^2} \right] \\
\times \frac{1}{\left[ F^2\sigma_M^2 (2 - \Gamma) \xi + \Gamma - 1 \right]} \frac{\left( 1 - M^2 - x^2 \right)^2 - (F - 1) M^2 \left( 1 - G^2 \right)^2}{G^2 \left( 1 - M^2 - x^2 \right)^2} \\
+ \frac{M^4 \sin^2(\theta)}{G^4} - \frac{M^2 \sin^2(\theta)}{G^4 \cos^2(\psi + \theta)} \left( 1 - x^2 \right)^{-1}. \tag{2.78}
\]

To obtain the equation at the Alfvén point, we make the following substitutions in the wind equation:

\[
G = 1, \tag{2.79}
\]
\[
M = (1 - x_A^2), \tag{2.80}
\]
\[
\left( \frac{dM^2}{d\theta} \right)_A = p_A, \tag{2.81}
\]
\[
\sigma_A = \frac{2x_A^2 \cos(\psi_A)}{p_A \sin(\theta_A) \cos(\theta_A + \psi_A)}, \tag{2.82}
\]
\[
\left( 1 - M^2 - x_A^2 \right)_A = \frac{1}{\sigma_A + 1}, \tag{2.83}
\]
\[
\left( \frac{1 - G}{1 - M^2 - x_A^2} \right)_A = x_A^2 \left( \frac{\sigma_A}{\sigma_A + 1} \right), \tag{2.84}
\]
\[
\left( \frac{G^2 - M^2 - x_A^2}{1 - M^2 - x_A^2} \right)_A = \frac{x_A^2 - \sigma_A (1 - x_A^2)}{x_A^2 (\sigma_A + 1)}, \tag{2.85}
\]

44
2.2 Extending the model

which leads to:

\[
0 = \mathcal{G}_A + 2 \frac{F - 2}{F^2 \sigma_M^2} \xi_A (\xi_A - 1) (1 - x_A^2) x_A^4 + \frac{\sin^2(\theta_A)(1-x_A^2)^2}{\cos^2(\psi_A + \theta_A)} \left[ (F - 1)(1-x_A^2) - 1 \right] + \frac{\mu^2 x_A^2 (F - 1) \sigma_A^2}{F^2 \sigma_M^2} \frac{1-x_A^2}{(\sigma_A + 1)^2} \left[ x_A^2 - \sigma_A (1-x_A^2) \right]^2 + 2 \frac{\cos(\psi_A) \sin(\theta_A) \sin(\psi_A + \theta_A)}{\cos^2(\psi_A + \theta_A)} x_A^2 (1-x_A^2)^2 \sigma_A + 1 \sigma_A. \tag{2.86}
\]

The above equation is the form we will use to determine \( p_A \) through \( \sigma_A \), so the gravitational addition to the ARC is simply the value of the gravity term at the Alfvén point. If we want to use equation (B6) in VK instead, we would have to add the following term to its right hand side:

\[
-\mathcal{G}_A \frac{F^2 \sigma_M^2 (\sigma_A + 1)^2}{\mu^2 x_A^2 1-x_A^2}. \tag{2.87}
\]

2.2.4 The kinetic gravity term

To include a gravity term into the equations, we need a prescription for it. A gravity term that only has the kinetic inertia, but neglects the thermal and electromagnetic inertias, is given in VTST. We will call it the kinetic gravity term, to contrast it with a gravity term that includes all these inertias. We can rewrite the gravity term found in VTST in a form compatible with VK and, making sure the scaling is correct, include it into the VK wind equation. We can also include a Paczyński-Wiita potential and will give the appropriate expressions as well.

Adding kinetic Newtonian gravity to the Alfvén Regularity Condition

By comparing the wind equations of VTST and VK, we can identify the gravity term in VTST. Using the definition of \( \kappa_{VTST} \) from VTST and the substitutions from table 2.2, the gravity term in VTST can now be rewritten in terms of VK:

\[
\kappa_{VTST} = \sqrt{\frac{GM}{\sigma_* V_*^2}} = \sqrt{\frac{GM \mu^2 x_A^4 (1-M^2-x_A^2)^2}{c^2 \sigma_A F^2 \sigma_M^2 (1-M^2-x_A^2)^2}}. \tag{2.88}
\]
Writing out the whole term found in the numerator of the VTST wind equation, we obtain the full kinetic gravity term:

$$G_{\text{kin}} = -\kappa_{\text{VTST}} \frac{\sin(\theta)}{G} = -\frac{GM}{c^2} \frac{\mu^2 x_A^4}{F^2 \sigma_M^2} \frac{1}{(1 - M^2 - x_A^2)^2} \sin(\theta).$$  (2.89)

Filling in the Alfvén values gives the gravity term addition to the ARC:

$$G_{\text{kin},A} = -\frac{GM}{c^2} \frac{\mu^2 x_A^4}{F^2 \sigma_M^2} \frac{1}{(1 - M^2 - x_A^2)^2} \frac{\sin(\theta_A)}{\sigma_A G}.$$  (2.90)

### Adding kinetic Paczyński-Wiita gravity to the Alfvén Regularity Condition

Since:

$$\frac{\sin(\theta)}{\sigma_A G} = \frac{1}{r},$$  (2.91)

and the Paczyński-Wiita potential is given by the substitution:

$$\frac{1}{r} \rightarrow \frac{1}{r - r_S} = \left[r - 2\frac{GM}{c^2}\right]^{-1},$$  (2.92)

where $r_S = 2GM/c^2$ is the Schwarzschild radius, the gravity term including the Paczyński-Wiita potential is given by:

$$G_{\text{kinPW}} = -\frac{GM}{c^2} \frac{\mu^2 x_A^4}{F^2 \sigma_M^2} \frac{1}{(1 - M^2 - x_A^2)^2} \left[\frac{\sigma_A G}{\sin(\theta)} - 2\frac{GM}{c^2}\right]^{-1}$$  (2.93)

$$= -\frac{\mu^2 x_A^4}{F^2 \sigma_M^2} \frac{1}{(1 - M^2 - x_A^2)^2} \left[\frac{c^2}{GM} \frac{\sin(\theta_A)}{\sigma_A G} - 2\right]^{-1}. $$  (2.94)

Filling in the Alfvén values gives the gravity term addition to the ARC:

$$G_{\text{kinPW},A} = -\frac{\mu^2 x_A^4}{F^2 \sigma_M^2} \frac{1}{(\sigma_A + 1)^2} \left[\frac{c^2}{GM} \frac{\sin(\theta_A)}{\sigma_A G} - 2\right]^{-1}. $$  (2.95)

### 2.2.5 The full gravity term

Another approach is to start with a general relativistic equation for gravity and see how the VK equations would change if we keep gravity, while still making the same approximations (Polko et al. 2013b). Although retaining gravity violates the assumptions used, we can check to what extent this happens afterwards. We will give a list of the equations that are modified by the inclusion of gravity.
To obtain a full gravity term, including the kinetic, thermal, and electromagnetic inertias, we have to modify both the energy equation and the transfield equation. We modify the energy equation based on an equation for $\mu$ that does depend on gravity. The transfield equation has an extra component accounting for gravity. This component is in the form of equations (A8) in VK, so we need to derive a new transfield equation from these equations in order to get the correct scaling for this new term. We will start with the energy equation.

A new equation for $\mu$

In general relativistic MHD (GRMHD) in the presence of gravity it is not $\mu$ that stays constant, but rather (Meier 2012):

$$\frac{(\mu - 1)c^2 + \mu \Phi}{c^2} = \text{constant} \equiv \mu' c^2 - c^2. \quad (2.96)$$

With

$$\Phi = -\frac{GM}{r} \quad (2.97)$$

the gravitational potential, it follows that

$$\frac{\Phi}{c^2} = -\frac{r_g}{r} = -\frac{G M \sin(\theta)}{c^2 \sigma_A G}, \quad (2.98)$$

and therefore at every point we should compute

$$\mu' = \mu \left[ 1 - \frac{G M \sin(\theta)}{c^2 \sigma_A G} \right] = \mu_A \left[ 1 - \frac{G M \sin(\theta_A)}{c^2 \sigma_A G} \right], \quad (2.99)$$

or, if we include a Paczyński-Wiita potential:

$$\mu' = \mu \left[ 1 - \frac{G M}{c^2} \left[ \frac{\sigma_A G}{\sin(\theta)} - 2 \frac{G M}{c^2} \right]^{-1} \right] = \mu_A \left[ 1 - \left[ \frac{c^2 \sigma_A G}{G M \sin(\theta_A)} - 2 \right]^{-1} \right]. \quad (2.100)$$

A new addition to $C_1$

Since the parameter $\mu$ is no longer constant, there is an additional term in the derivative of the modified energy equation. This term has no derivatives of $M^2$ or $\psi$ with respect to $\theta$, so it is added to the $C_1$ term:

$$C_1^+ = \left[ -2 \tan(\psi + \theta) G^6 F^2 \sigma_M^2 (1 - M^2 - x^2)^2 \sin^2(\theta) \right] - \left[ \frac{G M \sin(\theta)}{c^2 \sigma_A G} \right] \left[ 1 - \frac{G M \sin(\theta_A)}{c^2 \sigma_A G} \right] \times \left[ \frac{G^2 (1 - M^2 - x^2)^2 - x^2 (G^2 - M^2 - x^2)^2}{G^2 (1 - M^2 - x^2)^2} \right] \frac{\mu^2 x^2 A^2 \cos^2(\psi + \theta)}{F^2 \sigma_M^2 \sin^2(\theta)}. \quad (2.101)$$
The transfield equation

There also appears an additional gravitational force in the transfield equation. Since the transfield equation given in VK is not from the actual force equation, we need to derive a new transfield equation with the proper scaling. The centrifugal force term is given by

\[ f_{c_\perp} = \frac{B_0^2 \alpha F^{-2}}{4 \pi \varpi G^4} \left[ -\frac{x^4 \mu^2 x^2}{F^2 \sigma_M^2 M^2} \left( \frac{G^2 - M^2 - x^2}{1 - M^2 - x^2} \right)^2 \sin(\psi) \right], \quad (2.102) \]

the inertial force term by

\[ f_{I_\perp} = \frac{B_0^2 \alpha F^{-2}}{4 \pi \varpi G^4 \cos(\psi + \theta)} \left[ 2M^2 \sin^2(\theta) - \sin(\theta) \cos(\theta) \sin(\psi + \theta) \cos(\psi + \theta) \right. \\
+ \left. M^2 \sin^2(\theta) - M^2 \sin^2(\theta) \cos^2(\psi + \theta) \frac{\partial}{\partial \theta} \tan(\psi + \theta) \right], \quad (2.103) \]

the pressure force term by

\[ f_{p_\perp} = \frac{B_0^2 \alpha F^{-2}}{4 \pi \varpi G^4 \cos(\psi + \theta)} \left[ z^4 \frac{x^4}{F^2 \sigma_M^2} \left( - \frac{\Gamma - 1}{\Gamma} \frac{\xi(\xi - 1)}{M^2} \right) \\
- \frac{\Gamma - 1}{\xi - (\Gamma - 1)(\xi - 1)} \frac{\xi^2(\xi - 1)}{M^4} \sin(\psi + \theta) \cos(\psi + \theta) \frac{dM^2}{d\theta} \right], \quad (2.104) \]

the electric force term by

\[ f_{E_\perp} = \frac{B_0^2 \alpha F^{-2}}{4 \pi \varpi G^4 \cos(\psi + \theta)} \left\{ F x^2 \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} - x^2 \frac{\cos(\psi) \sin(\psi) \sin(\psi + \theta)}{\cos(\psi + \theta)} \right. \\
- \left. x^2 \sin^2(\theta) \frac{\partial}{\partial \theta} \tan(\psi + \theta) \right\}, \quad (2.105) \]

and the magnetic force term by

\[ f_{B_\perp} = \frac{B_0^2 \alpha F^{-2}}{4 \pi \varpi G^4 \cos(\psi + \theta)} \left\{ \frac{x^4 \mu^2 x^2}{F^2 \sigma_M^2} \left( \frac{1 - G^2}{1 - M^2 - x^2} \right)^2 \left( (F - 1) - \left[ \frac{1 - M^2 - x^2}{1 - G^2} \frac{dG}{d\theta} + \frac{dM^2}{d\theta} \frac{\sin(\psi + \theta) \cos(\psi + \theta)}{1 - M^2 - x^2} \right] \\
+ \left[ \frac{F \sin^2(\theta)}{\cos^2(\psi + \theta)} - \frac{\sin(\psi) \sin(\psi + \theta)}{\cos(\psi + \theta)} + \sin^2(\theta) \frac{\partial}{\partial \theta} \tan(\psi + \theta) \right] \right\}. \quad (2.106) \]
with the full transfield equation the sum of these parts:

\[ f_{C\perp} + f_{f\perp} + f_{p\perp} + f_{E\perp} + f_{E \perp} = 0. \]  

(2.107)

Casting the transfield equation above in the following form:

\[ A_2 \frac{dM^2}{d\theta} + B_2 \frac{d\psi}{d\theta} = C_2, \]  

(2.108)

yields the following determinant parts:

\[
A_2 = \left[ \frac{B_0^2 \sigma^{F-2}}{4\pi \alpha G^4 \cos(\psi + \theta)} \right] \frac{\xi^2 \chi^4_4}{F^2 \sigma^2_M} \sin(\psi + \theta) \cos(\psi + \theta) \]
\[
\left[ \frac{\mu^2 x^2 (1 - G^2)^2}{\xi^2 (1 - M^2 - x^2)^3} \frac{(\Gamma - 1) (\xi - 1)}{\xi - (\Gamma - 1) (\xi - 1) M^4} \right], \tag{2.109}
\]

\[
B_2 = \left[ \frac{B_0^2 \sigma^{F-2}}{4\pi \alpha G^4 \cos(\psi + \theta)} \right] \sin^2(\theta) \left[ \frac{(1 - x^2)}{\cos^2(\psi + \theta) - M^2} \right], \tag{2.110}
\]

\[
C_2 = \left[ \frac{B_0^2 \sigma^{F-2}}{4\pi \alpha G^4 \cos(\psi + \theta)} \right] \left\{ \frac{x_4^4 \mu^2 x^2}{F^2 \sigma^2_M} \left( \frac{G^2 - M^2 - x^2}{1 - M^2 - x^2} \right)^2 \frac{\sin(\psi) \cos(\psi + \theta)}{\sin(\theta)} \right. \]
\[
\left. + \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)} \left[ F - 2 - Fx^2 + x^2 + (1 + x^2) \frac{\cos(\psi) \sin(\psi + \theta)}{\sin(\theta)} \right] \right. \]
\[
\left. + \frac{x_4^4 \mu^2 x^2}{F^2 \sigma^2_M} \left( \frac{1 - G^2}{1 - M^2 - x^2} \right)^2 \left[ (F - 1) + \frac{2G^2}{1 - G^2} \frac{1 - M^2 - x^2}{1 - M^2 - x^2} \frac{\cos(\psi) \sin(\psi + \theta)}{\sin(\theta)} \right] \right. \]
\[
\left. + 2 \frac{\Gamma - 1}{F^2 \sigma^2_M} \frac{F - 2}{2} \frac{\xi (\xi - 1) x_4^4}{M^2} \right\}. \tag{2.111}
\]

The gravity term in the transfield equation

The gravitational force in the transfield equation is given by (Meier 2012):

\[
f_G = \left( \gamma \rho + \frac{\xi}{\xi^2} \right) (\nabla \Phi \cdot \hat{n}) = \left[ \frac{B_0^2 \sigma^{F-2}}{4\pi \alpha G^4 \cos(\psi + \theta)} \right] \]
\[
\times \left\{ \frac{x_4^4}{F^2 \sigma^2_M} \left[ \frac{\mu^2 (1 - M^2 - x^2)^2}{M^2 (1 - M^2 - x^2)^2} + \frac{\mu^2 x^2 (1 - G^2)^2}{2G^4 (1 - M^2 - x^2)^2} \right] \right. \]
\[
\left. - \frac{\Gamma - 1}{\Gamma} \frac{\xi (\xi - 1)}{M^2} \right\] \]
\[
+ \frac{1}{2} \frac{(1 + x^2) \sin^2(\theta)}{\cos^2(\psi + \theta)} \left[ \frac{GM \sin(\theta)}{c^2 \sigma_A G} \cos^2(\psi + \theta) \right]. \tag{2.112}
\]
The term above has the scaling of the physical transfield equation. To get it in the form of equation (B2e) we divide by $B_2^{F-2} \frac{\sin(\theta)}{4\pi c G^4 \cos(\psi + \theta)}$ to obtain:

$$
C_2^+ = \left\{ \frac{x^4}{F^2 \sigma_M^2} \left[ \frac{\mu^2 (1 - M^2 - x_A^2)^2}{M^2 (1 - M^2 - x^2)^2} + \frac{\mu^2 x^2 (1 - G^2)^2}{2G^4 (1 - M^2 - x^2)^2} \frac{\Gamma - 1}{M^2} \right] \right.

+ \frac{1}{2} \frac{(1 + x^2)}{\cos^2(\psi + \theta)} \left[ \frac{GM \sin(\theta)}{c^2 \sigma_A G} \cos^2(\psi + \theta) \right],

(2.113)

which, taking all the plusses and minuses into account, should also be the addition to $C_2$.

**The full gravity term with a Newtonian potential**

Due to the distributivity of summation, it is possible to calculate the gravity term separately from the rest of the numerator. This term is given by $C_1^+ B_2 - C_2^+ B_1$, which with:

$$
B_1 = \left[ -2 \tan(\psi + \theta) G^6 F^2 \sigma_M^2 (1 - M^2 - x^2)^2 \sin^2(\theta) \right] \frac{M^4}{G^4},

(2.114)
$$

and:

$$
B_2 = \left[ B_2^{F-2} \frac{\sin(\theta)}{4\pi c G^4 \cos(\psi + \theta)} \right] \left[ (1 - M^2 - x^2) + M^2 \sin^2(\psi + \theta) \right] \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)},

(2.115)
$$

turns into (neglecting the denominator of $C_1^+$ and the common factor of $B_1$ and $C_1^+$ and $B_2$ and $C_2^+$):

$$
G_{\text{full}} = - \frac{GM \sin(\theta)}{c^2 \sigma_A G} \left\{ \frac{\mu^2 x_A^4 (1 - M^2 - x_A^2)^2}{F^2 \sigma_M^2 (1 - M^2 - x^2)^2} (1 - x^2) \right.

- \frac{\mu^2 x_A^4 x^2 (G^2 - M^2 - x^2)^2}{F^2 \sigma_M^2 G^4 (1 - M^2 - x^2)^2} (1 - x^2) \right.

+ \frac{\mu^2 x_A^4 M^2 x^2 (G^2 - M^2 - x^2)^2}{F^2 \sigma_M^2 G^4 (1 - M^2 - x^2)^2} \cos^2(\psi + \theta) \right.

+ \frac{1}{2} \frac{\mu^2 x_A^4 M^4 x^2}{F^2 \sigma_M^2 G^4 (1 - M^2 - x^2)^2} \cos^2(\psi + \theta) \right.

- \frac{x_A^4}{F^2 \sigma_M^2} \frac{\Gamma - 1}{\Gamma} \xi(\xi - 1) M^2 \cos^2(\psi + \theta) \right.

+ \left. \frac{1}{2} \frac{M^4}{G^4 (1 + x^2) \sin^2(\theta)} \right\}.

(2.116)
If we neglect the thermal, magnetic and electric contributions, we are left with:

\[
\mathcal{R}_{\text{kin,rel}} = -\frac{GM}{c^2} \frac{\mu^2 x_A^4}{\sigma_A G} \frac{(1 - M^2 - x_A^2)^2}{F^2 \sigma_M^2} \left[ (1 - M^2 - x^2) + M^2 \sin^2 (\psi + \theta) \right]
\]

\[
- \frac{GM}{c^2} \frac{\mu^2 x_A^4}{\sigma_A G} \frac{(1 - M^2 - x_A^2)^2}{F^2 \sigma_M^2} (1 - M^2 - x^2)^2 M^2 \cos^2 (\psi + \theta)
\]

\[
= - \frac{GM}{c^2} \frac{\mu^2 x_A^4}{\sigma_A G} \frac{(1 - M^2 - x_A^2)^2}{F^2 \sigma_M^2} (1 - x^2). \tag{2.117}
\]

This differs from the kinetic gravity term in equation (2.89) only in the addition of \((1 - x^2)\), which goes to 1 as \(x^2\) is small in the non-relativistic limit.

**Adding full Newtonian gravity to the Alfvén Regularity Condition**

Since our wind equation has an additional gravitational term, this term evaluated at the Alfvén point should be added to the ARC as well. We set all parameters to their Alfvén values and substitute in equations (2.79 – 2.86) wherever a \(0/0\) occurs:

\[
\mathcal{R}_{\text{full,A}} = -\frac{GM}{c^2} \frac{\mu^2 x_A^4}{\sigma_A} \frac{1}{F^2 \sigma_M^2} \left\{ \mu_A^2 \frac{1 - x_A^2}{\sigma_A (\sigma_A + 1)^2} \left[ 1 - \frac{[x_A^2 - (1 - x_A^2)\sigma_A]^2}{x_A^2} \right] \right. \\
+ \left[ \mu_A^2 \frac{[x_A^2 - (1 - x_A^2)\sigma_A]^2}{x_A^2 (\sigma_A + 1)^2} (1 - x_A^2) + \frac{\mu_A^2 \sigma_A^2 (1 - x_A^2)^2}{2x_A^2 (\sigma_A + 1)^2} \right] \\
- \frac{\Gamma - 1}{\Gamma} \xi_A (\xi_A - 1) (1 - x_A^2) \\
+ \frac{1}{2} \frac{F^2 \sigma_M^2 (1 - x_A^2)^2 \sin^2 (\theta_A)}{x_A^4 \cos^2 (\psi_A + \theta_A)} \left[ 1 + x_A^2 \right] \cos^2 (\theta_A + \psi_A) \left\} \right. \tag{2.118}
\]
The full gravity term with a Paczyński-Wiita potential

The gravity term with a Paczyński-Wiita potential is given by:

\[
\mathcal{G}_{\text{fullPW}} = -\left(\frac{\mu^2 x_A^4}{F^2 \sigma_M^2} \frac{(1 - M^2 - x_A^2)^2}{(1 - M^2 - x_A^2)^2} (1 - x^2) \right)
\]
\[
- \frac{\mu^2 x_A^4}{F^2 \sigma_M^2} \frac{x^2 (G^2 - M^2 - x^2)^2}{G^4 (1 - M^2 - x^2)^2} (1 - x^2)
\]
\[
+ \frac{\mu^2 x_A^4}{F^2 \sigma_M^2} \frac{M^2 x^2 (G^2 - M^2 - x^2)^2}{G^4 (1 - M^2 - x^2)^2} \cos^2(\psi + \theta)
\]
\[
+ \frac{1}{2} \frac{\mu^2 x_A^4}{F^2 \sigma_M^2} \frac{M^4 x^2 (1 - G^2)^2}{G^4 (1 - M^2 - x^2)^2} \cos^2(\psi + \theta)
\]
\[
- \frac{x_A^4}{F^2 \sigma_M^2} \frac{\Gamma - 1}{\Gamma} \xi (\xi - 1) M^2 \cos^2(\psi + \theta)
\]
\[
+ \frac{1}{2} \frac{M^4}{G^4} (1 + x^2) \sin^2(\theta) \right) \left[ \frac{c^2 \sigma_A G}{\mathcal{G} \sin(\theta)} - 2 \right]^{-1}. \tag{2.119}
\]

Adding full Paczyński-Wiita gravity to the Alfvén Regularity Condition

The addition to the ARC of a gravity term with a Paczyński-Wiita potential is then given by:

\[
\mathcal{G}_{\text{fullPW,A}} = -\left[ \frac{c^2 \sigma_A}{\mathcal{G} \sin(\theta)} - 2 \right]^{-1} \cdot \frac{x_A^4}{F^2 \sigma_M^2} \left[ \mu_A^2 \frac{1 - x_A^2}{(\sigma_A + 1)^2} \left[ 1 - \frac{[x_A^2 - (1 - x_A^2) \sigma_A]^2}{x_A^2} \right] \right]
\]
\[
+ \left[ \mu_A^2 \frac{[x_A^2 - (1 - x_A^2) \sigma_A]^2}{x_A^2 (\sigma_A + 1)^2} (1 - x_A^2) + \frac{\mu_A^2 \sigma_A^2 (1 - x_A^2)^2}{2x_A^2 (\sigma_A + 1)^2} \right]
\]
\[
- \frac{\Gamma - 1}{\Gamma} \xi (\xi - 1) (1 - x_A^2)
\]
\[
+ \frac{1}{2} \frac{F^2 \sigma_M^2}{x_A^4} (1 - x_A^2) \sin^2(\theta_A) \right] \cos^2(\theta_A + \psi_A) \right) \left[ \frac{c^2 \sigma_A G}{\mathcal{G} \sin(\theta)} - 2 \right]^{-1}. \tag{2.120}
\]

2.3 Numerical method for finding solutions

In this section we will describe the numerical method we have employed to find solutions to the equations, from the calculations required to start the integration, via the integration steps, to the iteration towards a solution that crosses both the MSP and MFP.

52
2.3 Numerical method for finding solutions

2.3.1 Initial setup

We start by specifying values for the parameters $F$, $\Gamma$, $\theta_A$, $\psi_A$, $x_A$, $\sigma_M$, $q$ explained in section 2.1.5, and the mass of the compact object $M$, and the physical radius of the Alfvén point $\sigma_A$. Since these parameters are completely degenerate, we combine them by expressing $\sigma_A$ in gravitational radii.

Using the equation for $M$ at the Alfvén point ($M^2_A = 1 - x^2_A$), we can calculate the specific enthalpy at the Alfvén point:

$$M^2 = \frac{q}{(\xi - 1)^{1/(\Gamma - 1)}}, \quad (2.121)$$

using a cubic root solver. This approach works when $\Gamma$ has values $4/3$ or $5/3$, but not in the general case. Next we apply the appropriate expression of the Alfvén regularity condition to calculate $p_A$ using the Newton-Raphson technique. With this value we can calculate:

$$\sigma_A = \frac{2x_A^2 \cos(\psi_A)}{p_A \sin(\theta_A) \cos(\theta_A + \psi_A)}, \quad (2.122)$$

and consequently:

$$\mu^2 = \frac{(\sigma_A + 1)^2}{x_A^2 - \left[\frac{x_A^2}{\sigma_A} - (1 - x_A^2)\right]^2} \left[\frac{x_A^2 \xi_A^2}{x_A^2 \cos^2(\theta_A + \psi_A)} + \frac{F^2 \sigma_M^2 \left(1 - x_A^2\right)^2 \sin^2(\theta_A)}{x_A^2 \sin^2(\theta_A) \cos(\theta_A + \psi_A)}\right], \quad (2.123)$$

to finally obtain $\mu'$ using either equation (2.99) or (2.100). Now we have all the required values to start off the integration.

2.3.2 Integration step

We use the Runge–Kutta method with Cash–Karp coefficients, modified to continue until the integration step becomes zero, to solve simultaneously the differential equation for the cylindrical radius of the field line:

$$\frac{dG^2}{d\theta} = \frac{2G^2 \cos(\psi)}{\sin(\theta) \cos(\psi + \theta)}, \quad (2.124)$$

and the wind equation with the appropriate gravity term, using $\theta$ as our independent variable. From the new values of $\theta$, $G$, and $M$ we can calculate:

$$x = x_A G, \quad (2.125)$$
\( \mu \) from equation (2.99) or (2.100), \( \xi \) from equation (2.121), and \( \psi \) from the energy equation:

\[
\frac{\mu^2 G^4 (1 - M^2 - x_A^2)^2 - x^2 (G^2 - M^2 - x^2)^2}{\xi^2 G^4 (1 - M^2 - x^2)^2} = 1 + \frac{F^2 \sigma_\nu^2 M^4 \sin^2(\theta)}{\xi^2 x^2 \cos^2(\psi + \theta)}.
\] (2.126)

With these values we can initiate the next integration step until the integration fails due to zero step size.

### 2.3.3 Iteration towards a solution

Once we have a full integration of a field line, there are in general four possible results. At both the MSP and MFP either the denominator of the wind equation crossed (or tends to \(2\)) zero before the numerator, or the numerator crossed zero before the denominator. Based on these results we change the fitting parameters \(x_A^2\) (for the MFP) and \(q\) (for the MSP) until the other crossed first, signifying a double crossing, or a smooth transition of the corresponding singular point in between those two values of the fitting parameter (see figure 2.3). We have found it convenient to change \(x_A^2\) until an MFP is found, then change \(q\) until an MSP is found, and then repeat. After two such steps it is possible to interpolate from these four points in order to get a better estimate of the parameter values giving a solution. It is also possible to use a linear combination of \(x_A^2\) and \(q\) to follow the MFP and MSP line and converge to a solution faster, though in cases where these lines are almost parallel, this approach may be less stable. The diagnostic plot shown in figure 2.3 can be used to quickly sample a large range in \(x_A^2\) and \(q\), and visually estimate the correct parameter values.

---

\(^2\)If the value of the denominator becomes small while the numerator is still large, the integration reaches zero step size due to the resulting large acceleration, so in general the denominator does not cross zero.
2.3 Numerical method for finding solutions

Figure 2.3: Diagnostic plot to help locate solutions. The four colours correspond to the four possible results of an integration. Yellow: the numerator crosses zero first at both the MFP and MSP. Red: the denominator crosses first at the MFP and the numerator at the MSP. Green: the denominator crosses first at the MFP and the numerator at the MSP. Blue: the denominator crosses first at the MFP and MSP. The line between the yellow/green and red/blue area denotes parameter values which smoothly cross the MFP, the line between the yellow/red and green/blue area denotes parameter values which smoothly cross the MSP. Where these two lines cross lies a solution with both an MFP and MSP. The big block size is due to finite sampling of the $x^2 A q$-plane.
Determining the optimal locations for shock acceleration in magnetohydrodynamical jets

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Abstract

Observations of relativistic jets from black hole systems suggest that particle acceleration often occurs at fixed locations within the flow. These sites could be associated with critical points that allow the formation of standing shock regions, such as the magnetosonic modified fast point (MFP). Using the self-similar formulation of special relativistic magnetohydrodynamics by Vlahakis & Königl, we derive a new class of flow solutions that are both relativistic and cross the MFP at a finite height. Our solutions span a range of Lorentz factors up to at least 10, appropriate for most jets in X-ray binaries and active galactic nuclei, and a range in injected particle internal energy. A broad range of solutions exists, which will allow the eventual matching of these scale-free models to physical boundary conditions in the analysis of observed sources.

3.1 Introduction

Jets have been observed around a large variety of astrophysical objects, such as young stellar objects (YSOs), accreting white dwarfs, X-ray binaries (XRBs), and active galactic nuclei (AGN), and are also thought to drive gamma-ray bursts (GRBs). In
YSOs, jets facilitate angular momentum transport, allowing the central star to accrete more matter, and likely play a similar role in accreting black hole systems, where in AGN they are thought to also affect the evolution of their host galaxy (e.g. Best et al. 2006). While there are enormous differences between the types and scales of objects around which jets can occur, the origins of jets seem to be remarkably similar, requiring the basic ingredients of infalling/collapsing, rotating matter, and magnetic fields.

Despite these seemingly simple initial conditions, there are currently many outstanding problems in our understanding of jets, from their creation to their matter content and internal physics. One important facet of jets observationally is their hallmark synchrotron emission dominating the radio bands in particular, extending up to at least the near infrared (NIR) in XRBs (Fender et al. 1997). At higher frequencies, the typical power-law spectral energy distributions (SEDs) are generally interpreted as optically thin synchrotron emission from accelerated particles, both in AGN (Marscher & Gear 1985) and in XRBs (Fender 2002). An effective way to accelerate radiating particles into a power-law distribution is via diffusive shock acceleration (e.g. Bell 1978; Drury 1983) off scattering centres in turbulent plasma flows. Once initiated, this process must be distributed throughout the flow to account for the lack of spectral ageing over vast distances along the jets (e.g. Jester et al. 2001).

But where does the acceleration itself begin, and what triggers it? There is increasing evidence that the start of this region is offset from the central compact object. For example in the jet of the AGN M87, recent VLBI observations show the synchrotron emission starting at a region offset by $\sim 100 r_g$ from the core (Junor et al. 1999; Walker et al. 2008). Similarly, the start of power-law acceleration in compact jets would be indicated by a transition from optically thick emission, with a flat/inverted spectral index, to an optically thin power law at a distinct location in the SED. Such a break has been observed directly so far only in one source, the Galactic XRB GX 339-4, in the NIR (Corbel & Fender 2002), during the “hard” or “non-thermally dominated” accretion state associated with compact jet formation (see state definitions in, e.g., McClintock & Remillard 2006). Because of the stratification of emission regions in compact jets (e.g. Blandford & Konigl 1979), the lower the frequency where this turnover occurs, the further the location along the jet where particle acceleration starts. In XRBs, a break in the NIR corresponds to an offset of $\sim 10 - 1000 r_g$, and models of the broadband data of most black hole XRBs in the hard state so far seem to require such a break (e.g. Markoff et al. 2001, 2005; Nowak et al. 2005; Migliari et al. 2007; Maitra et al. 2009a).

The fact that the start of the acceleration region seems to occur at roughly the same location in several systems could be indicative of a critical point occurring in a magnetohydrodynamical (MHD) flow, particularly the magnetosonic modified fast
3.2 Method

3.2.1 Background

During the acceleration and collimation of jets, magnetic fields are thought to efficiently extract rotational energy from either the compact object (Blandford & Znajek point (MFP). At the MFP, the collimating magnetic field lines turn inward toward the jet axis, potentially leading to recollimation shocks, while at the same time the flow becomes causally disconnected so shocks can occur without disrupting the flow upstream. Such a shock region thus would occur at a fixed location in the flow, closely connected to the MFP, and would be an ideal location for particle acceleration to begin. We wish to investigate the feasibility of this premise in this paper.

Because of the complexity involved in relativistic MHD including strong gravity, many groups are using the results of simulations to study the formation and development of jets. These simulations often show the development of a steady outflow in which the magnetic field is remarkably self-similar and axisymmetric near the launch point (see, e.g., figures 2 and 11 in McKinney 2006). However to study specifically the development of critical points in the flow and their dependence on external boundary conditions, current MHD simulations either do not extend far enough from the black hole or if they do, they are too computationally expensive. Assuming that the resulting flows retain a self-similar structure, at least when gravity does not dominate as indicated in the simulations, we adopt the formalism developed by Vlahakis & Königl (2003a, hereafter VK03). By assuming axisymmetry and a self-similar field line geometry, VK03 reduce the exact equations of special relativistic MHD to a one-dimensional problem. Although VK03 focused on jets in GRBs, this treatment is also applicable to other MHD jets such as in AGN and microquasars.

In an earlier self-similar treatment that was non-relativistic, Vlahakis et al. (2000) presented a solution where the flow crosses the MFP at a finite height above the disk. In VK03, however, they only found relativistic solutions with an MFP occurring at infinity (meaning that the flow asymptotically approaches a perfect cylindrical geometry). In this paper we extend the study of VK03 and derive new solutions in which the relativistic flow crosses the MFP at a finite location above the disk. In section 2, we describe the VK03 model and our method for exploring the full parameter space of solutions. In section 3, we present the first relativistic solutions that pass through an MFP. In section 4, we discuss our results, and the dependence of the MFP location on the model parameters. We also describe how this work sets the stage for further development to connect the flow to regions near the disk where gravity can no longer be ignored. Section 5 contains our conclusions.
Determining the optimal locations for shock acceleration in magnetohydrodynamical jets

1977) or the accretion disk (Blandford & Payne 1982). The latter models are in the Newtonian limit, with the matter considered cold, meaning there is negligible thermal pressure causing bulk acceleration to non-relativistic velocities.

These cold, non-relativistic solutions were generalised to the relativistic regime by Li et al. (1992), allowing the bulk velocity to attain relativistic speeds. VK03 further extended the solutions to include the “hot” regime, allowing the random motions of the particles to become relativistic and thus the jets to be hydrodynamically accelerated even at the base, where temperatures are high. It is this last scenario that we base this work upon.

Starting from the equations of time-dependent special relativistic MHD, we make the following assumptions to render them more tractable: ideal MHD, no gravitational field or external force (and thus self-similarity holds), axisymmetry, a zero azimuthal electric field and time independence. Following the terminology of VK03, after scaling the equations to make them non-dimensional, we are left with two coupled differential equations (equations (3.8) and (3.9) in the appendix). Combining these two coupled differential equations, we obtain a single equation for $\frac{dM^2}{d\theta}$ (equation (3.13a), with $M$ being the Alfvénic Mach number and $\theta$ the angle of the point on the field line with the axis of symmetry) which acts like a “wind equation”, much akin to the wind equation of the Parker solar wind model (Parker 1958). This wind equation, along with the other algebraic equations (see the appendix), can be solved for the velocity, magnetic and electric field strength, density and pressure along a field line. Due to the self-similar assumption, once solutions are obtained for one field line, all other field lines can be obtained by simple scaling. An example of this self-similarity and the meaning of some of the parameters used can be found in figure 3.1.

Instead of the single critical (or sonic) point of the Parker solar wind model, due to the inclusion of magnetic fields, the obtained wind equation has three locations where the denominator crosses zero. Starting from the accretion disk, these are the modified slow point (MSP), the Alfvén point, and the MFP. The MSP and MFP are also called the slow and fast magnetosonic separatrix surfaces. The Alfvén point is the location where the relativistic collimation speed of the flow toward the axis ($V_\theta$) is given by:

$$ (\gamma V_\theta)^2 = \frac{B_0^2 (1 - x^2)}{4\pi \rho_0 \xi}, \quad (3.1) $$

where $\gamma = 1 / (1 - V^2 / c^2)^{1/2}$ is the Lorentz factor, $B$ is the strength of the magnetic field, $x$ is the cylindrical radius in units of the light cylinder radius, $\rho_0$ is the baryon rest-mass density, and $\xi c^2$ is the specific relativistic enthalpy (the variables are described in more detail in section 3.2.2). The denominator of $dM^2 / d\theta$ with the Alfvén
point divided out can be expressed as:

\[
\mathcal{D} = \left( \frac{\gamma V_\theta}{c} \right)^4 - \left( \frac{\gamma V_\theta}{c} \right)^2 \left[ \frac{U_S^2}{c^2} + \frac{B^2 - E^2}{4 \pi \rho_0 c^2} \right] \\
+ \frac{U_S^2}{c^2} \frac{B_\theta^2 (1 - x^2)}{4 \pi \rho_0 c^2},
\]

(3.2)

with:

\[
U_S^2 = c^2 \frac{(\Gamma - 1)(\xi - 1)}{(2 - \Gamma)\xi + \Gamma - 1},
\]

(3.3)

c the velocity of light, \(E\) the strength of the electric field, and \(\Gamma\) the polytropic index. The MSP and MFP are, by definition, the locations where \(\mathcal{D} = 0\) (VK03).

At every critical point, the numerator of \(dM^2/d\theta\) should also pass through zero to ensure a smooth crossing. This translates into a regularity condition at the critical points and fixes the value of a free parameter. Even though the MSP should be crossed smoothly to obtain a solution that describes the entire jet from the accretion disk to the termination point, gravitational effects cannot be ignored at the MSP. Since the equations do not include gravity (as it is not compatible with the self-similarity assumption in relativistic flow), we do not try to fit for the MSP. Therefore we fit two critical points, and correspondingly two parameters are fixed, in our approach described below, \(p_A\) for the Alfvén point and \(\sigma_M\) for the MFP.

The physical importance of the MFP is that it is the location where not even the fastest signals can travel upstream anymore, meaning anything downstream from the MFP is causally disconnected from the region upstream. If, for example, a shock were to exist beyond the MFP, it could not disrupt the flow leading to that shock, allowing it to be a permanent feature. As mentioned above, at the MFP there is a component of the velocity heading toward the axis. This can lead to a collimation shock shortly beyond the MFP, causing the magnetic energy to be converted to particle energy and the jet to become kinetically dominated. Another possibility is that the flow remains magnetically dominated, and, after reaching a minimum radius, bounces back, retaining an ordered magnetic field (see, e.g., Contopoulos & Lovelace 1994). This is not in conflict with the statement in VK03 that the only physically acceptable case in the super-Alfvénic regime is for the flow to become asymptotically cylindrical, as this statement only applies to solutions with \(F > 1\).

### 3.2.2 Model Parameters

The prescription we are following from VK03 has nine free parameters that determine the solution, whose effects are described below. Following the same notation, a Roman subscript \(A\) signifies the value of a variable at the Alfvén point and an italic \(A\) denotes a value with respect to the poloidal magnetic flux function.
3 Determining the optimal locations for shock acceleration in magnetohydrodynamical jets

Figure 3.1: Sketch of self-similar field lines projected onto the meridional plane. For any \( \theta \) the values of the variables describing the field line are exactly the same, only scaled by their respective distances.

Free Parameters

The exponent \( F \) determines the current distribution. A value \( F > 1 \) corresponds to the current-carrying regime, with higher values of \( F \) ensuring faster collimation, but if \( F < 1 \) we are in the return-current regime. The restriction on \( F \) is that it cannot be negative: \( F > 0 \). Although we consider \( F \) to be a free parameter, for this paper we chose to keep it fixed at 0.75.

The adiabatic index \( \Gamma \) can have values of \( 4/3 \) for relativistic and \( 5/3 \) for non-relativistic solutions.

\( \theta_A \) gives the angle where the Alfvén point is located with respect to the axis of symmetry. It is limited by the value of \( \psi_A \): \( \theta \in (90^\circ - \psi_A, 90^\circ] \).

\( \psi_A \) gives the poloidal slope of the field line at the Alfvén point with respect to the accretion disk. This in turn is limited by \( \theta_A \): \( \psi_A \in (90^\circ - \theta_A, 90^\circ] \).

\( x_A^2 \) is the radius squared of the Alfvén point in terms of the light cylinder radius. For \( x_A^2 \to 1 \) the solution becomes more force free. The allowed values are \( x_A^2 \in (0, 1) \).

\( \sigma_M \) is the magnetisation parameter in the monopole solution of Michel (1969) and is related to the mass-to-magnetic flux ratio. The constraint is \( \sigma_M > 0 \).

\( q \) is the dimensionless adiabatic coefficient and is constant along a field line. For a large value of \( q \), the specific relativistic enthalpy of the matter is high, for \( q \to 0 \) equation (3.6) shows \( \xi \to 1 \) and the flow is cold. Therefore \( q \geq 0 \).
3.2 Method

Figure 3.2: Three-dimensional plot of solutions with an MFP, with parameters $F = 0.75$, $\Gamma = 5/3$, $x_A^2 = 0.75$, and $\theta_A = 50^\circ$. On the front side, the blue and red surfaces are connected. The lower surface does not extend all the way to the $\psi_A$-axis for all values of $\psi_A$. Because we favour solutions with strong magnetic fields, we focus on the upper surface, with high $\sigma_M$.

$p_A$ is the derivative of $M^2$ with respect to the polar angle $\theta$ at the Alfvén point. Accelerating flow implies $p_A < 0$.

$B_0\sigma_0^{2-F}$, with reference magnetic field $B_0$ and reference length $\sigma_0$, is the scaling of the solution, relating the dimensionless values to physical dimensions. We do not yet apply our solutions to specific black hole systems, so this parameter is not used here but it will be important for future applications of our solutions.

The smooth crossing of the Alfvén point is ensured by calculating $p_A$ from the corresponding regularity condition, given by equation (3.12). In the same way we determine $\sigma_M$ by crossing the MFP, although this parameter sometimes can have two values (see figure 3.2). This leaves $F$, $\Gamma$, $\theta_A$, $\psi_A$, $x_A^2$, and $q$ (and $B_0\sigma_0^{2-F}$) to satisfy the boundary conditions at the source and, indirectly, at the end of the jet.

Other Parameters and Variables

There are also parameters derived from the above values,

$\mu$ determines the total energy-to-mass-flux ratio ($\mu c^2$) and is conserved along a field line. This parameter is determined from equation (3.11) and $\mu > 1$.

$\sigma_A$ is the value of the magnetisation function $\sigma$, defined as the Poynting-to-matter energy flux, at the Alfvén point. This parameter is determined from equation (3.10). As $\mu$ cannot be negative, from equation (3.11) follows $\sigma_A \in [0, \frac{\Delta_A}{1-x_A}]$.

And finally we describe the other variables used in the equations.

$\xi$ determines the specific (per baryon mass) relativistic enthalpy ($\xi c^2$). If $\xi = 1$
Determind the optimal locations for shock acceleration in magnetohydrodynamical jets

we have cold, pressureless matter. $\xi$ will drop from the high temperatures at the base of the jet as matter is mainly accelerated hydrodynamically and at some point above the disk drop down to 1. From this point on all acceleration is magnetic. This variable is determined from equation (3.6).

$M$ is the Alfvénic Mach number, the velocity of the flow in terms of the Alfvén velocity.

$G$ is the radius in terms of $x_A$ and is therefore equal to 1 at the Alfvén point.

### 3.2.3 Numerical Method

To find solutions with an MFP, we obtained expressions for $dM^2/d\theta$, given by equation (3.13a), and for $d\phi/d\theta$ by combining the derivative of the energy equation with the transfield equation using the determinant method. Because $d\phi/d\theta$ is very unstable near the Alfvén point, as the numerator has a first order zero point there and the denominator a second order one, we reverted to the energy equation, given by equation (3.7), to determine $\phi$. To start off the integration from the Alfvén point, we specify $F$, $\Gamma$, $\theta_A$, $\psi_A$, $q$ and an initial guess for $\sigma_M$. We determine $p_A$ from the Alfvén regularity condition, equation (3.12). Integrating outward from the Alfvén point, we determine whether the numerator or denominator crosses zero first and adjust $\sigma_M$ accordingly until both cross at the same time. We then proceed to explore the range of solutions which cross both the Alfvén point and MFP (see figure 3.3).

For plotting purposes, we have divided out the factor $x^6(1 - M^2 - x^2)^2$ in both the numerator and denominator.

As discussed above, because there is no gravity in the model, we do not try explicitly to cross the MSP. Gravitational effects should play a large role close to the black hole, and the self-similar equations cannot predict accurately where the MSP is located.

### 3.2.4 Approach

To begin our exploration of parameter space, we chose a solution from Vlahakis et al. (2000) known to have an MFP (specified below their figure 4) with parameters $x = 0.75$, $\gamma = 5/3$, $\theta_s = 60^\circ$, $\psi_s = 45^\circ$, $\kappa^2 = 15$, $\mu = 10.9239$, $\lambda^2 = 2.7935$, $p_s = -5.5744$, and $e = 9.4487$. By comparing terms in Vlahakis et al. (2000) and VK03, it is possible to translate these parameters to the parameters used in VK03. Because we are using the relativistic equations from VK03, the parameters of our first solution (see table 3.1) differ slightly from the corresponding parameters above and we vary $\sigma_M$ to obtain a critical solution again. From this solution we were able to traverse parameter space, while allowing only critical solutions with an MFP. By increasing $x_A^2$, we were able to obtain higher velocities of the jet. After achieving

64
3.2 Method

Figure 3.3: Solution c. For parameters see table 3.1. The scaling for the y-axis is a combination of a linear and logarithmic part, using the function $\text{sign}(x) \log_{10}[1 + \text{abs}(x)/10^{-12}]$. The vertical black line gives the location of the Alfvén point (at $\theta \approx 0.87$). Even though the numerator (red line) and denominator (green line) change sign, it can be seen from their ratio (blue line, $dM^2/d\theta$) that the MFP is crossed smoothly. We ceased the integration shortly after the MFP.

relativistic velocities, we focused on finding solutions with higher values of $q$, as the solutions so far were cold. But for a fixed value of $x^2_\Lambda$, there is a maximum value of $q$ that produces critical solutions crossing the MFP. This is due to the fact that the collection of all solutions forms a surface in the multidimensional parameter space, and we had reached a maximum for $q$ for the fixed values of the other parameters (see figure 3.2).

The surfaces of valid solutions have roughly the same appearance for the explored range of $\theta_\Lambda$ and $x^2_\Lambda$. To describe the effect of $\theta_\Lambda$ and $x^2_\Lambda$ on the solutions, we approximate the graph as a cone with the base in the $\psi_\Lambda, \sigma_M$-plane and with the maximally allowed value of $q$ as its height. If we increase $x_\Lambda$, the base of the cone shrinks while moving to higher $\sigma_M$, and the height decreases. The latter limits the maximum value that $x^2_\Lambda$ can be increased to. If we decrease $\theta_\Lambda$, the height increases, indirectly allowing higher values for $x^2_\Lambda$. The area of the base becomes bigger and shifts to higher $\psi_\Lambda$ and $\sigma_M$, with the upper surface becoming steeper. Due to the shape of the surface, it is possible to increase any two parameters of $x^2_\Lambda$, $q$, and $\sigma_M$ at the expense of the third.

By extending our search to three parameters ($\psi_\Lambda$, $q$, and $\sigma_M$), we were able to move around this point and continue increasing $q$. This revealed a multidimensional
3 Determining the optimal locations for shock acceleration in magnetohydrodynamical jets

surface that is double valued in $\sigma_M$. An example of this surface is shown in figure 3.2, which also includes our most relativistic solution presented below, solution c. The numerator, denominator, and acceleration of $M^2$ in this latter solution are also plotted in figure 3.3.

3.3 Results

In this section, we present the various solutions crossing the MFP, from the first one that we found to one with relativistic temperature and bulk flow we sought, while describing the features particular to a certain solution. The parameters of our solutions are given in table 3.1 and the main properties in figure 3.4. As we are not yet applying our solutions to specific black hole systems, we do not use the scaling parameters $B_0\omega_0^{-F}$ and $\omega_0$.

**Table 3.1: Parameters of Solution**

<table>
<thead>
<tr>
<th>Solution</th>
<th>$F$</th>
<th>$\Gamma$</th>
<th>$x_A^2$</th>
<th>$q$</th>
<th>$\theta_A$ (deg)</th>
<th>$\psi_A$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0.75</td>
<td>5/3</td>
<td>4.7676 $\times 10^{-3}$</td>
<td>2.5348 $\times 10^{-6}$</td>
<td>60</td>
<td>45</td>
</tr>
<tr>
<td>b</td>
<td>0.75</td>
<td>5/3</td>
<td>0.75</td>
<td>2.5 $\times 10^{-6}$</td>
<td>60</td>
<td>45</td>
</tr>
<tr>
<td>c</td>
<td>0.75</td>
<td>5/3</td>
<td>0.75</td>
<td>0.12</td>
<td>50</td>
<td>55</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Solution</th>
<th>$\sigma_M$</th>
<th>$p_A$</th>
<th>$\sigma_A$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>2.14228 $\times 10^{-4}$</td>
<td>-5.54314</td>
<td>5.42665 $\times 10^{-3}$</td>
<td>1.01197</td>
</tr>
<tr>
<td>b</td>
<td>1.01241</td>
<td>-1.50363</td>
<td>3.14708</td>
<td>5.44084</td>
</tr>
<tr>
<td>c</td>
<td>2.53981</td>
<td>-1.66651</td>
<td>2.6039</td>
<td>9.85117</td>
</tr>
</tbody>
</table>

Note.—The values for the upper six parameters ($F$ through $\psi_A$) are exact, for the lower four ($\sigma_M$ through $\mu$) they are rounded off.

3.3.1 Solution a: A Cold, Slow Jet

This solution is the closest to the non-relativistic parameter values given in Vlahakis et al. (2000) that conforms to the Alfvén regularity condition (ARC), which is the transfield equation at the Alfvén point. The solution crosses the MFP at $\theta \approx 0.15$ rad or 8.6° ($G \approx 15.3$). The Alfvén point is located at $\theta \approx 1.05$ rad or 60°. The top left panel of figure 3.4 gives the meridional projection of the magnetic field lines. The jet overcollimates after a maximum radius of almost 16 times the Alfvén radius at $\theta \approx 0.20$ rad or 11.5°, shortly before the MFP. The second left panel of figure 3.4
Figure 3.4: Properties of solutions. Row 1 shows the geometry of the field line, where the height has the same scaling as the cylindrical radius. Row 2 shows the Lorentz factor $\gamma$, the Lorentz factor multiplied by the specific relativistic enthalpy and the Poynting-to-mass flux ratio ($S = -\omega \Omega B_\phi / \Psi A c^2$). These last two add up to the total energy-to-mass flux ratio $\mu$, which is a constant along the field line. Row 3 shows the causal connection opening angle $\arcsin(1/\gamma)$, and the opening half-angle of the outflow, which is negative after overcollimation. Row 4 shows the squares of the Alfvénic Mach number ($M^2$), and the radius in units of the light cylinder radius ($x^2$).

shows that the flow is cold throughout ($\xi - 1 \ll 1$) due to the very small value for $q$ and low $x_A^2$. As $M^2 = 1 - x_A^2$ at the Alfvén point, equation (3.6) gives a value for $\xi_A$ very close to 1. The energy of the matter $\xi$ (including the dominant rest mass energy) is much higher than the energy in the magnetic field throughout. Therefore, even though the magnetic acceleration is efficient, the jet is not accelerated to relativistic velocities ($\gamma < 1.02$). The third centre panel shows the “causal connection” opening angle $\arcsin(1/\gamma)$ and the opening half-angle of the outflow, which goes from 60° to a few degrees overcollimation. Although the causal connection opening angle has little importance for non-relativistic flows, as it remains very close to 90° since $\gamma \sim 1$, it is shown for completeness.
3.3.2 Solution b: A Cold, Fast Jet

After our first solution, we increased the velocity of our jet by increasing $x_A^2$. As the top centre panel in figure 3.4 shows, after a long period where the field line remains almost parabolic, the jet in this solution overcollimates as well. This is caused by magnetic hoop stresses and may allow a shock region to develop beyond the MFP. The Alfvén point is again located at $\theta \approx 1.05$ rad and the MFP at $\theta \approx 0.063$ rad or $3.6^\circ$ ($G \approx 145$). The second centre panel shows the Lorentz factor and the enthalpy of the flow. The flow here also is seen to be cold ($\xi \approx 1$), meaning the jet is mainly magnetically accelerated, which is again due to the small value of $q$. The Poynting-to-mass flux ratio ($S \equiv -\pi \Omega B_\phi / \Psi A c^2$) decreases, showing magnetic energy being transferred into kinetic energy, with the flow reaching a Lorentz factor of 2.8. The bottom centre panel shows the squares of the light cylinder radius, $x \equiv \pi \Omega / c$, and the Alfvénic Mach number, $M$. When $x^2 = 1$ the light surface is reached, which is the radius where the field circular velocity reaches the speed of light.

3.3.3 Solution c: A Warm, Very Fast Jet

After having achieved a relativistic solution for the cold plasma case, we would like to find solutions with an increased flow temperature. To do so requires increasing the value of $\xi$. As $M^2$ is given by $1 - x_A^2$ at the Alfvén point, equation (3.6) shows that increasing $x_A^2$ and/or $q$ has the desired effect. Unfortunately, for larger $x_A^2$ the maximum value of $q$ decreases. By choosing a lower value for $\theta_A$ the attainable values for $x_A^2$ and $q$ are increased, leading to a warmer flow. This solution is shown in the third column of figure 3.4. The Alfvén point is located at $\theta \approx 0.87$ rad or $50^\circ$ and the MFP at $\theta \approx 0.041$ rad or $2.4^\circ$ ($G \approx 87.4$). Near the beginning of the flow $\xi \approx 2.9$.

The Lorentz factor of the flow at the MFP is 8.3, which is mainly due to the high initial Poynting flux.

It can be seen in the second right panel of figure 3.4 that $\xi$ always drops to 1 and from the fourth right panel that $M^2$ always dominates $x^2$ near the MFP. As the Lorentz factor is given by:

$$\gamma = \frac{\mu}{\xi} \frac{1 - M^2 - x_A^2}{1 - M^2 - x^2},$$

(3.4)

the final Lorentz factor is approximately $\mu$. This means that while the jets may start as Poynting flux dominated, eventually they convert most of their Poynting flux and become kinetic energy dominated.

3.3.4 Location of the MFP

Since we are interested in the location of the MFP, we would like to know how it depends on the model parameters. As it is challenging to sample the full parameter
space, we will focus on the region around the solution closest to observed jets, solution \( c \). By allowing all parameters to vary and looking at the effect this has on the location where the MFP occurs, we can draw the following conclusions: the MFP moves outward (smaller \( \theta \)) when the Alfvén point occurs at a smaller angle, (lower \( \theta_A \)), when the temperature at the base of the flow is increased (higher \( q \)), when the flow at the Alfvén point is already close to collimation (higher \( \psi_A \)), or when the Alfvén point moves closer to the light cylinder radius, making the flow more force free (higher \( x_A \)).

### 3.4 Discussion

We have succeeded in obtaining new, solutions for a relativistic, magnetised flow that smoothly crosses the MFP at a finite height (\( \theta > 0 \)) above the system equator. These solutions suggest that it should be possible to construct better, more MHD-consistent jet models where the location of the acceleration region is determined a priori from the physical boundary conditions.

So far none of the solutions derived remains Poynting flux dominated up to the MFP, which is probably due to the relatively small value of \( \sigma_M \) found so far. Increasing \( x_A^2 \) and especially decreasing \( \theta_A \) will allow higher values of \( \sigma_M \) to be used. The same can also be done at the expense of \( q \).

We also have described some of the relations between the different parameters (see section 3.2.2) and the effect they have on each other. Having a more force-free solution (higher \( x_A^2 \)) decreases the allowed range of temperatures (\( q \)) and collimation at the Alfvén point (\( \psi_A \)), but at the same time allows a higher value of \( \sigma_M \) that provides a critical solution. Keeping all other parameters fixed, there is a maximum value of \( x_A^2 \) for which a solution is possible at all. This maximum may be increased by moving the Alfvén point closer to the disk (smaller \( \theta_A \)). This change has the additional effects of allowing a broader range for the collimation angle at the Alfvén point (\( \psi_A \)) while at the same time shifting this range toward higher collimation. It also allows for a higher temperature of the flow (\( q \)) or a higher magnetic field strengths (\( \sigma_M \)). Any pair of parameters \( x_A^2, q, \) and \( \sigma_M \) may be increased at the expense of the third.

Two parameters not varied so far are \( F \) and \( \Gamma \). Higher values for \( F \) should ensure faster collimation and might therefore be very important for the exact location of the MFP. Similarly, we may want to explore the \( \Gamma = 4/3 \) case for jets with extremely relativistic temperature. However, for the weaker jets in AGN and XRBs that we plan to target, the radiating particle distributions are generally thought to peak at mildly relativistic energies.
3.5 Conclusion

If the start of the particle acceleration region in steady jets is indeed associated with the magnetosonic fast critical point in the bulk flow, then our results support the conclusion that such a region could occur at a fairly stable location about the launch point. All of the solutions found overcollimate shortly before the MFP, as would be expected for the initiation of shock development. By starting with a solution that crosses the MFP in the non-relativistic case of Vlahakis et al. (2000), we were able to extend the solution through the multidimensional parameter space toward relativistic velocities and temperatures, while retaining the critical point. This feature sets our results apart from the work of VK03, whose formalism we adopted, in that they are immediately applicable to observed compact jet sources with an optically thick-to-thin break in the synchrotron spectrum, such as hard state XRBs and weakly accreting AGN. Our most promising solution is a jet outflow with mildly relativistic temperature, and Lorentz factor of $\sim 10$, also appropriate for the steady jets in both XRBs as well as AGN.

It is clear that a wide range of parameter space is left still unexplored, that can be exploited for matching physical boundary conditions appropriate to known astrophysical sources. However, before a radiative model can be constructed around the dynamical “backbone” provided by the solutions presented here, a prescription for including gravity must be included, to extend these solutions through the MSP and allow connection with a physical model of the accretion flow/corona. We are currently working on matching these necessarily non-self-similar solutions to those presented here, which will be presented in a separate work. Once this solution is in place, we will have a much more physically consistent model (compared to, e.g., Markoff et al. 2005) to use in the fitting of data from accreting black holes across the mass scale, which show compact, steady jets. As there seems to be no shortage of possible solutions, we are confident that we can match physical boundary conditions with critical solutions.

3. A Equations

Here, we list the equations we have used for reference. See section 3.2.2 for a description of the parameters and variables. The radius in units of the light cylinder radius is given by:

$$x = x_A G. \quad (3.5)$$

The Alfvén Mach number is given by:

$$M^2 = q \frac{\xi}{(\xi - 1)^{1/(\Gamma - 1)}}. \quad (3.6)$$
The energy equation, also called the Bernoulli equation, is given by:

\[
\frac{\mu^2 G^4 (1 - M^2 - x_\lambda^2)^2 - x_\lambda^2 (G^2 - M^2 - x_\lambda^2)^2}{G^4 (1 - M^2 - x_\lambda^2)^2} = 1 + \frac{F^2 \sigma_M^2 M^4 \sin^2 (\theta)}{\xi^2 x^4 \cos^2 (\psi + \theta)}. \tag{3.7}
\]

The differential equation for the radius is:

\[
\frac{dG^2}{d\theta} = \frac{2G^2 \cos (\psi)}{\sin (\theta) \cos (\psi + \theta)}. \tag{3.8}
\]

The transfield equation is given by:

\[
G \sin^2 (\theta) \frac{d}{d\theta} \left[ \tan (\psi + \theta) \frac{1 - M^2 - x^2}{G} \right] = (F - 1) \frac{\chi^4_M \mu^2 x^2}{F^2 \sigma_M^2} \left( \frac{1 - G^2}{1 - M^2 - x^2} \right)^2 \\
- \sin^2 (\theta) M^2 + F x^2 - F + 1 \\
- \frac{\chi^4_A \mu^2 x^2}{F^2 \sigma_M^2} \left( \frac{G^2 - M^2 - x^2}{1 - M^2 - x^2} \right)^2 \\
+ 2 \frac{\Gamma - 1 - F - 2 \xi (1 - x^4)}{\Gamma} \frac{\xi}{F^2 \sigma_M^2} \frac{M^2}{M^2}. \tag{3.9}
\]

The magnetisation function \( \sigma \) and the fractions at the Alfvén point are given by:

\[
\sigma_\lambda = \frac{2 \chi^2_M \cos (\psi_\lambda)}{\rho_A \sin (\theta_\lambda) \cos (\theta_\lambda + \psi_\lambda)}, \quad \left( \frac{1 - M^2 - x^2}{1 - M^2 - x^2} \right)_\lambda = \frac{1}{\sigma_\lambda + 1}, \quad \left( \frac{1 - M^2 - x^2}{1 - M^2 - x^2} \right)_M = \frac{x^2_A - (1 - x^2_\lambda) \sigma_\lambda}{x^2_A (\sigma_\lambda + 1)}. \tag{3.10}
\]

We obtain \( \mu^2 \) by inserting these relations into the energy equation:

\[
\mu^2 = \frac{(\sigma_\lambda + 1)^2}{x^2_A - \left[ x^2_A - \sigma_\lambda (1 - x^2_\lambda) \right]^2} \left[ x^2_A \xi_\lambda^2 + \frac{F^2 \sigma_M^2 (1 - x^2_\lambda)^2 \sin^2 (\theta_\lambda)}{x^2_A \cos^2 (\theta_\lambda + \psi_\lambda)} \right]. \tag{3.11}
\]

The Alfvén regularity condition is obtained by substituting these relations into the transfield equation:

\[
\frac{F^2 \sigma_M^2 (1 - x^2_\lambda) (\sigma_\lambda + 1)^2 \sin (\theta_\lambda)}{\mu^2 \cos^2 (\theta_\lambda + \psi_\lambda)} \left\{ - \frac{\Gamma - 1}{\Gamma} \frac{(F - 2)(\xi_\lambda - 1) (1 - x^2_\lambda)}{x^2_A \xi_\lambda^2} \sin (\theta_\lambda) \\
+ 2 \cos (\psi_\lambda) \sin (\theta_\lambda + \psi_\lambda) \frac{\sigma_\lambda + 1}{\sigma_\lambda} + \frac{\sin (\theta_\lambda)}{x^2_A} \left[ (F - 1) (1 - x^2_\lambda) - 1 \right] \right\}
\]

\[
= \left[ x^2_A - \sigma_\lambda (1 - x^2_\lambda) \right]^2 - (F - 1) \sigma_\lambda^2 (1 - x^2_\lambda) \\
- 2 \frac{\Gamma - 1}{\Gamma} (F - 2) \frac{\xi_\lambda - 1}{\xi_\lambda} \left\{ x^2_A - \left[ x^2_A - \sigma_\lambda (1 - x^2_\lambda) \right]^2 \right\}. \tag{3.12}
\]
The wind equation is given by:

\[
\frac{dM^2}{d\theta} = \frac{C_1 B_2 - C_2 B_1}{A_1 B_2 - A_2 B_1},
\]

(3.13a)

where:

\[
A_1 = -2x^4 \cos^2(\psi + \theta)(1 - M^2 - x^2)\left\{ \mu^2 x_2 A M^2 (1 - G^2)^2 \right. \\
\left. + G^2 (1 - M^2 - x^2) \left\{ \frac{F^2 \sigma_M^2 M^2 \sin^2(\theta)}{x^4 \cos^2(\psi + \theta)} - \frac{\xi(T+1)q_{T-1}}{M^2 \left[ \frac{G^2}{(T-1)} + 1 \right]} \right\} \right) 
\]

(3.13b)

\[
B_1 = -2F^2 \sigma_M^2 M^4 \sin^2(\theta) \tan(\psi + \theta) G^2 (1 - M^2 - x^2)^2, 
\]

(3.13c)

\[
C_1 = x^4 \cos^2(\psi + \theta) \mu^2 \frac{dG^2}{d\theta} \left[ - \left( 1 - M^2 - x_2^2 \right)^2 + 2x_2^2 \left( G^2 - M^2 - x^2 \right) (1 - x_2^2) \right] \\
+ \frac{dG^2}{d\theta} \left( 1 - M^2 - x^2 \right) \left( 1 - M^2 - 3x^2 \right) \left[ \xi^2 x^4 \cos^2(\psi + \theta) + F^2 \sigma_M^2 M^4 \sin^2(\theta) \right] \\
+ 2F^2 \sigma_M^2 M^4 \sin^2(\theta) G^2 (1 - M^2 - x^2)^2 \left[ \cos(\theta) - \frac{1}{G^2} \frac{dG^2}{d\theta} + \tan(\psi + \theta) \right]. 
\]

(3.13d)

\[
A_2 = -\sin^2(\theta) \tan(\psi + \theta), 
\]

(3.13e)

\[
B_2 = \sin^2(\theta) \frac{1 - M^2 - x^2}{\cos^2(\psi + \theta)}, 
\]

(3.13f)

\[
C_2 = -\sin^2(\theta) \frac{1 - M^2 - x^2}{\cos^2(\psi + \theta)} + \sin^2(\theta) \tan(\psi + \theta) \left[ x_2^4 \frac{dG^2}{d\theta} + \left( 1 - M^2 - x^2 \right) \frac{1}{G^2} \frac{dG}{d\theta} \right] \\
+ (F-1) \frac{x_2^4 \mu^2 x_2^2}{F^2 \sigma_M^2} \left( \frac{1 - G^2}{1 - M^2 - x^2} \right)^2 - \sin^2(\theta) \frac{M^2 + F x^2 - F + 1}{\cos^2(\psi + \theta)} \\
- \frac{x_2^4 \mu^2 x_2^2}{F^2 \sigma_M^2} \left( \frac{G^2 - M^2 - x^2}{1 - M^2 - x^2} \right)^2 + 2 \frac{\Gamma - 1}{\Gamma} \frac{F - 2 \xi (\xi - 1) x^4}{F^2 \sigma_M^2 M^2}. 
\]

(3.13g)
Abstract

We present a new, approximate method for modelling the acceleration and collimation of relativistic jets in the presence of gravity. This method is self-similar throughout the computational domain where gravitational effects are negligible and, where significant, self-similar within a flux tube. These solutions are applicable to jets launched from a small region (e.g., near the inner edge of an accretion disk). As implied by earlier work, the flow can converge onto the rotation axis, potentially creating a collimation shock.

In this first version of the method, we derive the gravitational contribution to the relativistic equations by analogy with non-relativistic flow.

This approach captures the relativistic kinetic gravitational mass of the flowing plasma, but not that due to internal thermal and magnetic energies. A more sophisticated treatment, derived from the basic general relativistic magnetohydrodynamical equations, is currently being developed.

Here we present an initial exploration of parameter space, describing the effects the model parameters have on flow solutions and the location...
of the collimation shock. These results provide the groundwork for new, semi-analytic models of relativistic jets which can constrain conditions near the black hole by fitting the jet break seen increasingly in X-ray binaries.

4.1 Introduction

Despite decades of study, astrophysical jets are still an enigmatic component of our universe. Understanding the details of their physics would advance fields as diverse as star formation, gamma-ray bursts (GRBs), galaxy evolution and cluster dynamics. Jets in all of these objects probably share three basic ingredients: a source of energy and particles such as an accretion flow, magnetic fields created or carried inward by the flow, and rotation. However, the exact details of how these elements combine to launch and collimate jets in the observed systems is not at all clear.

Black holes are ideal test cases to understand jet physics because of their enormous range in mass, allowing us to study jet dynamics over an equally large range of time-scales. Accreting stellar mass black holes in a binary with a companion star (BHBs) have jets that can undergo an entire launch and quench cycle on time-scales of months, which is related at least in part to the accretion state of the disc (Esin et al. 1997; Meier 2001; Fender et al. 2004; McClintock & Remillard 2006). In their supermassive cousins, active galactic nuclei (AGN), such transitions would be expected to occur on time-scales of millions to a billion times longer. To assess the extent to which such a mass-scaling operates, we must have a better understanding of which elements of jet physics persist despite a dramatic range of environment and scales.

One key feature of jets regardless of source seems to be their ability to accelerate particles to highly relativistic energy. We observe the outcome of these accelerated particles directly via optically thin synchrotron emission in, e.g., AGN (Marscher & Gear 1985) and in BHBs (Fender 2001). When the jets are compact enough to become self-absorbed, the superposition of many synchrotron-emitting components along the length of the jet leads to a flat or slightly inverted total spectrum (Blandford & Konigl 1979) until the point where the jet emission becomes entirely optically thin. At this point the spectral index will steepen dramatically to $\alpha \sim 0.5 - 0.8$ typically, where $\alpha$ is defined such that $F_\nu \propto \nu^{-\alpha}$. This break in the spectrum can be physically associated with the most compact region in the jets where particle acceleration is present. In low-luminosity AGN jets this break typically occurs in the GHz range (Ho 1999), and if jets are self-similar across the black hole mass range, in BHBs such a break would be predicted to occur in the infrared (IR) bands (Markoff et al. 2001, 2003; Heinz & Sunyaev 2003). Because the spectral break also indicates the total radiative power of the jets, and plays a key role in driving mass-scaling predictions
such as the Fundamental Plane of black hole accretion (Merloni et al. 2003; Falcke et al. 2004; Plotkin et al. 2012b), it is a pivotal parameter linking the inner jets (and assumedly conditions near their launch point) to the outer jets associated with strong radio through IR emission. Deriving this link and overall jets structure in an MHD-consistent way is the focus of this paper.

Constraints on the location of the jet break can come from both observations as well as from spectral fitting. Because the optical/IR (OIR) bands are often dominated by a stellar companion or the accretion disc, the optically thick-to-thin break has been observed explicitly in only one BHB so far, GX 339-4 (Corbel & Fender 2002; Gandhi et al. 2011; see also Migliari et al. 2006 for a similar break for neutron stars), and has been indirectly constrained for Cygnus X-1 based on recent MIR observations (Rahoui et al. 2011). These observations confirm the theoretically predicted break location, and the recent boom in multiwavelength monitoring of many BHBs is leading to new studies of the break and its scaling with luminosity (Russell et al., in prep.). The location in frequency of the break can also be rather tightly constrained by fitting multiple spectra of BHBs and AGN in states associated with compact jets, which has the advantage of also associating a size scale for the region, and distance from the black hole. A series of works have consistently found the region of the jet where particle acceleration initiates seems to be offset from the black hole at distances of $\sim 10$–1000 $r_g$, depending on source luminosity, where $r_g$ is the gravitational radius, $GM/c^2$ (Markoff et al. 2001, 2003, 2005, 2008; Migliari et al. 2007; Gallo et al. 2007; Maitra et al. 2009a). Accordingly, this zone also should be associated with an offset in the start of the synchrotron core from the black hole in direct imaging. Such a measurement is very difficult to do because of the spatial resolution required. At least for the jet in M87, a relatively nearby AGN (17.0 ± 0.3 Mpc, Tonry et al. 2001) with a large supermassive black hole (6.4 ± 0.5 × 10$^9$ M$\odot$, Gebhardt & Thomas 2009, although note this value is twice the mass found in previous studies), the offset is $\sim 30$–100 $r_g$ (Junor et al. 1999; Walker et al. 2008; Hada et al. 2011). Interestingly, recent astrometry for M87 has also shown that the location of the radio core with respect to the black hole is extremely stable over several years (Asada et al. 2011). A natural explanation for such a region for the start of particle acceleration would be a shock where continual diffusive acceleration occurs (e.g. Bell 1978; Drury 1983). Because particles need to be continually accelerated, the shock should be a steady feature within the compact jets.

In self-similar, axisymmetric MHD flows, the bulk flows can be accelerated through several singular points in order: the modified slow point (MSP), the Alfvèn point (AP), and the modified fast point (MFP) (Blandford & Payne 1982, hereafter BP82). At the MFP, the velocity of the flow greatly exceeds the fast magnetosonic speed, which is the fastest that signals in the jet can travel. Thus beyond the MFP the flow is
causally disconnected from the flow closer to the central object, allowing disruptions such as shocks to form. A stable location for the MFP in a given flow would therefore provide a natural explanation for a steady feature associated with particle acceleration. This location could, however, be a function of mass and accretion power and vary from source to source within a certain range. In the models mentioned above, the shock location is a fitted free parameter that suggestively falls into the same range for a variety of sources. However, to understand if this is a physically meaningful result, we should ideally be able to show that this location corresponds to a feature that can be derived self-consistently within a theoretical framework, and tied to conditions at the jet launch point. The earlier models are based on a hydrodynamical (HD) velocity profile (Falcke & Biermann 1995; Falcke & Markoff 2000), and thus do not invoke the role of magnetic fields other than as a global parameter. However, if we move to a MHD framework, we can derive the exact location of the MFP (and assumedly the shock) based on boundary conditions defined by the launch point of the jets, and then test if the theory provides the correct scalings as observed, and as seem to be driving the Fundamental Plane.

In this paper we take a semi-analytical approach to derive new flow solutions (a new jet model) with the explicit aim of exploring the conditions for formation of the MFP, by solving the equations of relativistic MHD under the assumptions of self-similarity and axisymmetry. These assumptions greatly simplify the equations describing the jets, and are consistent with the results of numerical MHD simulations, which almost always produce a magnetic field line geometry near the launch point that is remarkably self-similar and axisymmetric (see, e.g., figures 2 and 11 in McKinney 2006).

We need to make one further approximation to construct our solutions: in order to link the properties of the MFP to the conditions at the base of the jet, we need to solve for the AP and MSP in a regime which could potentially be very close to the black hole where gravity becomes important. The MSP is a natural point to associate with the launch region of the jets, similar to the sonic nozzle in hydrodynamical flows, and is thus also the place where conditions could eventually be matched to the accretion inflow, such as a magnetic corona or radiatively-inefficient accretion flow (RIAF; Narayan & Yi 1994; Blandford & Begelman 1999; Quataert & Gruzinov 2000; Merloni & Fabian 2002). However, gravity will play a significant role so close to the black hole; therefore, it also needs to be included in order to correctly derive and cross the MSP. Until now no semi-analytical formalism has been developed to describe a relativistic flow passing through all three singular points, because in a relativistic framework gravity is not self-similar. Therefore in this paper, in order to accomplish this connection, we apply a bridging solution between our previous relativistic self-similar flow model without gravity (Polko et al. 2010, hereafter chapter
4.2 Method

In order to tie the conditions at the start of the particle acceleration region to the conditions at the base of the jet, we need to have a full description of a hot, relativistic flow. Because close to the central object gravity is as important as temperature and the magnetic field strength, the gravitational potential must be taken into account in order to be able to describe the jet in this region. Because this region is generally subrelativistic or mildly relativistic, we start with a solution that includes gravity in a non-relativistic flow. Far from the black hole the flow will be relativistic, but gravity will no longer be important, so we can use the equations derived in chapter 3 to describe this flow. We then seek solutions that satisfy both the relativistic flow without gravity and the non-relativistic flow with gravity at opposite ends of the jet.
4.2.1 A physical description of the flow

In the framework of self-similar relativistic MHD, we find that jets in our solutions are accelerated in the following way: starting from the base of the flow, the initial bulk acceleration is provided through thermal pressure gradients by means of sound waves by the very hot particles surrounding the central object. When the flow exceeds the sound speed, this mechanism becomes inefficient, and acceleration due to the centrifugal force of the rotating magnetic field by means of Alfvén waves takes over. After the flow surpasses the Alfvén speed, magnetic pressure becomes the dominant mechanism. When the flow velocity increases beyond the fast magnetosonic speed, the final boost is given by the pinching of the magnetic field. If at this point the flow overcollimates, a shock may form, which can accelerate the particles into the observed power-law distributions, thus constituting the start of the particle acceleration region.

In prior work (see chapter 3) we were unable to probe the region where the initial bulk acceleration occurs due to the absence of gravity in our formalism. However, the singular MSP can be produced only by the inward gravitational force balancing the outward magnetic and thermal pressure forces. Thus by including gravity we can describe the jet from very close to the central object to the first instance of overcollimation, providing the connection between the conditions at these two points. We will discuss how we deal with the issues regarding self-similarity in §4.4.

4.2.2 A new solution technique: the $C^\infty$-continuous bridging method

The matching of two flow solutions valid in different regimes of parameter or physical space is a common technique in physics. Some techniques simply solve the two different equations in the different regimes and then match the solutions at a single (and arbitrary) point. Such bridging solutions are only $C^0$-continuous, meaning its derivative is not a continuous function. Other methods may involve spline fitting the two solutions together in a finite bridging region. In this case, the two solutions are valid in each of their respective regimes, but in the bridging region neither is strictly valid. Instead, the spline fit is an approximation of the transition between the two solutions, and it is $C^1$-continuous or better.

Our method for bridging the non-relativistic VTST00 MHD wind equation with gravity and the relativistic VK03 one without gravity is similar to the above approaches. That is, when gravitational forces are important and the flow non-relativistic, we obtain the VTST00 MHD solution. However, when gravity is no longer important, the flow has the character of the VK03 relativistic solutions, whether it is relativistic or not. The difference here is that, because of the nearly-identical structures of the VTST00 and VK03 equations, we can perform this task with the construction of a
4.2 Method

The method, then, will be \( C^\infty \)-continuous, with all derivatives being continuous functions – a distinct advantage when one is dealing with equations which have singular points through which the solution must pass.

4.2.3 The basic \( C^\infty \)-continuous method

We now give a brief overview of our previous work in chapter 3 and describe how we here extend it by bridging it with a non-relativistic formulation including gravity valid close to the black hole.

In chapter 3 we combined the energy equation (describing the forces along a field line) and the transfield equation (describing the forces perpendicular to a field line), to construct a wind equation, which describes the bulk acceleration of the flow and fully specifies the jet geometry. Using this single differential equation, it was possible to obtain jet solutions that crossed an MFP at a finite distance from the origin. The terms in the wind equation thus obtained using the formalism of VK03 turned out to be almost identical to the terms in the wind equation given in VTST00. Apart from relativistic corrections to all the corresponding terms, the only difference is the gravity term in the VTST00 wind equation, which is non-relativistic. If we modify the VK03 wind equation to include gravity, our combined wind equation will reduce exactly to the formalism of VTST00 when relativistic effects apart from gravity are negligible as is the case close to the black hole, while it also reduces exactly to the formalism of VK03 when gravity becomes negligible as is the case far away from the black hole. Between these two regimes the wind equation describes the bridging regime where relativistic effects may begin to dominate the gravitational effects. Since all these regimes are described by one continuous equation, the resulting solution will be a smooth transition between the two regimes. We prefer this approach over matching the solutions of the two regimes at an arbitrary location in the jet, while trying to satisfy any number of continuity conditions.

Since the notation in VTST00 differs from VK03, we use equations (8)–(12) in VTST00 and equations (24) in VK03 to translate the notation to that used in the latter. This conversion also provides the relativistic corrections to the gravitational mass. Using this relativistic velocity in the definition of the gravity term in VTST00 yields a relativistic form of the gravity term, allowing us to include it into the VK03 wind equation. The term by term comparison of the two wind equations ensures that we use the proper scaling. Since some of our solutions get very close to the central black hole, we adapted the gravity term to include a pseudo-Newtonian potential (Paczyński & Wiita 1980).
After these steps (see appendix 4.A) the gravity term has the following form:

\[
- \frac{\mu^2 x_A^4}{F^2 \sigma_M^2} \frac{(1 - M^2 - x_A^2)^2}{(1 - M^2 - x_A^2)^2} \left[ \frac{c^2 \sigma_A G}{G M \sin(\theta)} - 2 \right]^{-1},
\]  

(4.1)

where the subscript A denotes values at the Alfvén point and \( \mu c^2 \) is the total energy-to-mass flux ratio, \( x_A \) is the cylindrical radius distance of the AP scaled to the light cylinder radius, \( F \) controls the current distribution, \( \sigma_M \) is the magnetisation parameter, \( M \) is the Alfvénic Mach number, \( x \) is the cylindrical radius distance scaled to the light cylinder radius, \( c \) is the speed of light, \( \sigma_A \) is the cylindrical radius distance to the AP, \( G \) is the cylindrical radius distance scaled to the AP, \( G \) is the gravitational constant, \( M \) is the mass of the black hole and \( \theta \) is the spherical polar angle (see table 4.2 for an overview of all model parameters).

The first two fractions consist purely of constants along a given field line, providing the overall scaling. The third fraction is proportional to \( (\gamma \xi) c^2 \) (see 4.3), where \( \gamma \) is the Lorentz factor and \( \xi c^2 \) is the specific relativistic enthalpy (per baryon mass). The last fraction corresponds to \( (r/r_g - 2)^{-1} \), with \( r \) the spherical radius and \( r_g \) the gravitational radius, and it is this term that represents the pseudo-Newtonian potential.

Several equations besides the wind equation are needed to fully determine a solution, so we have made sure that we take into account all equations in which the gravity term appears. As explained above, gravity, as described by equation (4.1), shows up in the numerator of the wind equation. By evaluating the wind equation at the AP, we obtain the Alfvén regularity condition (ARC, see equation 4.6), which also depends on gravity. The ARC allows us to calculate the slope of \( M^2 \) at the AP, \( p_A \), one of the initial parameters of the integration. The gravity term, as it appears on the right hand side of the ARC in VK03, is given by:

\[
- \frac{x_A^2}{1 - x_A^2} \left[ \frac{c^2 \sigma_A}{G M \sin(\theta_A)} - 2 \right]^{-1}.
\]  

(4.2)

Another location in which the gravity term may appear is in the energy equation evaluated at the AP. This equation is used to determine \( \mu \), another initial parameter of the integration (see equation 4.7). However, by evaluating the energy equation of VTST00 at the AP, the gravity term in that equation vanishes there. So equation (B5) in VK03 is still valid (see appendix 4.C). The transfield equation is incorporated into the wind equation, but not used independently in our calculations, so its dependence on gravity is irrelevant here.

By making these changes we have been able to find solutions that can pass through all three singular points of a hot, relativistic MHD flow, extending all models so far published (for a qualitative overview see table 4.1).
4.2 Method

Table 4.1: Overview of self-similar, axisymmetric MHD models, classified whether they include gravity, allow for a warm flow, allow relativistic velocities, and cross the MSP, AP and MFP, respectively.

<table>
<thead>
<tr>
<th>Paper</th>
<th>Gravity</th>
<th>Warm</th>
<th>Relativistic</th>
<th>MSP</th>
<th>AP</th>
<th>MFP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Li et al. (1992)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Vlahakis et al. (2000)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Vlahakis &amp; Königl (2003a)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Polko et al. (2010)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Polko et al. (this work)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

4.2.4 The effects of including gravity on the solutions

In every instance of gravity the ratio $\sigma_A/M$ appears. Thus for a central object that is twice as massive, if the AP is moved twice as far from the axis of symmetry, the solution remains unchanged. This result is a direct consequence of mass-scaling, since all properties are expressed in gravitational radii. Regulating the gravitational force exerted by the compact object can therefore be achieved by changing either $M$ or $\sigma_A$ and, apart from the overall physical size of the system, these two actions are equivalent. Increasing the effect of gravity can thus be thought of either as increasing the mass of the central object (increasing the reach of the gravitational well), or as decreasing the radius of the AP (moving the Alfvén point deeper into the gravitational well).

One way to show these effects, would be by starting with a singular solution containing both an MFP and an MSP, slowly increasing $\sigma_A$ to decrease the effect of gravity, while keeping solutions with an MFP and MSP. Unfortunately, these solutions do not smoothly transition into solutions that only have an MFP. It is therefore impossible to directly compare these new solutions crossing all three singular points to solutions without an MSP and we are left noting the effects within these new solutions.

The significance of having a solution with an MSP is that only then it is possible to connect conditions at the MFP to conditions very close to the central object. Even though the extension of a field line in linear distance may not be much, we can be sure that the results in the sub-Alfvénic regime have meaning by satisfying a strict boundary condition. We equate the MSP with the launch point of the jet and fit the conditions at the MSP to the accretion flow. Hence, extending the solutions to the MSP allows a link between the accretion flow and the start of the particle acceleration region, which we will use in later work.

Compared with the Paczyński-Wiita potential, the Newtonian potential has a
4 Linking accretion flow and particle acceleration in jets. I. New relativistic magnetohydrodynamical jet solutions including gravity

Table 4.2: List of model parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>Sets the radial dependence of the magnetic field strength</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>Adiabatic index</td>
</tr>
<tr>
<td>$x_A$</td>
<td>Cylindrical radius in terms of the light cylinder radius</td>
</tr>
<tr>
<td>$\sigma_M$</td>
<td>Magnetisation parameter</td>
</tr>
<tr>
<td>$q$</td>
<td>Dimensionless adiabatic coefficient</td>
</tr>
<tr>
<td>$\sigma_A$</td>
<td>Cylindrical radius</td>
</tr>
<tr>
<td>$M$</td>
<td>Mass of the central object</td>
</tr>
<tr>
<td>$\theta_A$</td>
<td>Angle with respect to the axis of symmetry</td>
</tr>
<tr>
<td>$\psi_A$</td>
<td>Angle of the field line with respect to the disc</td>
</tr>
<tr>
<td>$p_A$</td>
<td>Derivative of the Alfvénic Mach number squared w.r.t. $\theta$</td>
</tr>
<tr>
<td>$\sigma_A$</td>
<td>Magnetisation func. (Poynting-to-matter energy flux ratio)</td>
</tr>
<tr>
<td>$\mu c^2$</td>
<td>Total energy-to-mass flux ratio</td>
</tr>
<tr>
<td>$\xi c^2$</td>
<td>Specific (per baryon mass) relativistic enthalpy</td>
</tr>
</tbody>
</table>

Notes: The subscript $A$ means the value of the variable at the AP. For a complete description of the parameters, please see sections 2.1.5 and 3.2.2.

weaker gravitational force at the same radius. As it is the gravitational force that balances all other forces at the MSP, using a Newtonian potential would move the MSP closer to the black hole, possibly beyond the event horizon for certain solutions. To avoid this, we adopt the Paczyński-Wiita potential to approximate the gravitational potential close to the black hole. The adaptation from a Newtonian potential is straightforward (see appendix 4.A).

4.2.5 Model parameters

By including gravity in the equations, two additional parameters need to be specified before an integration. These are the mass of the central object, $M$, and the cylindrical radius of the AP, $\sigma_A$. As mentioned above, only the ratio of these parameters appears in the equations, so these two parameters are linearly dependent and solutions with the same ratio are indistinguishable if all other parameters are fixed. Since a solution is given in gravitational radii, this means that any solution can be scaled up or down to any physical size. For our initial solution we have chosen to set $M$ to the mass of a typical stellar mass black hole of $10 M_\odot$ (although this value will eventually be set to the deduced mass of an observed black hole system) and to set $\sigma_A$ to $5 r_g$. We will
look at the effects of mass scaling at a later time.

The parameters $F$ and $\Gamma$ are set to a single value for now, but can be changed if necessary. For example, since $F$ influences the geometry of the jet, it also affects the collimation. By allowing it to vary, we will have the freedom to adjust the degree of collimation to fit the broad-band data on our sources. Due to the self-similar nature of the equations, the height of the MFP will depend on the initial radius at the disc. Because we want a representative location, we choose to anchor the field line in the most active part of the accretion disc close to the central object. Within the innermost stable circular orbit (ISCO) the accreting matter can no longer form a disc and will move towards the central object as a radiatively inefficient flow. Therefore we set $F = 0.75$, which corresponds to a wide range of accretion disc models, including the Blandford-Payne (BP82), Shakura-Sunyaev (Shakura & Sunyaev 1973), and radiatively-inefficient accretion flow (RIAF) models. We also set $\Gamma = 5/3$, appropriate for non-radiation-dominated jets in the hard state, where the radiation and the particles do not behave as a single fluid.

The choice of parameters to fit the singular points will affect the solution we find. Were we to use $\theta_A$ and $\psi_A$ to fit for the MFP and MSP positions, the angle of the Alfvén point and the slope of the field line there would be varied to find a singular solution. If, on the other hand, $x_A^2$ and $q$ were used to fit for the MFP and MSP, the light cylinder radius (and hence angular velocity) and dimensionless adiabatic coefficient would be varied instead. This last combination most closely approaches self-similarity out of all the pairs of parameters that we have studied. Therefore, in this work we will use $x_A^2$ and $q$ as fitting parameters. It is important to note that by choosing different fitting parameters only the way we move through parameter space changes, not the set of solutions themselves.

### 4.3 Results

#### 4.3.1 First solution

Starting with the warm (initial $\xi \approx 4.7$) and fast (final $\gamma \approx 10$) solution $c$ in chapter 3, we use our new method to explore parameter space to determine a solution with an MSP. The parameter values of this new solution are very close to those of solution $c$ (see table 4.3), establishing the relative ease with which solutions can be found. Figure 4.1 shows the numerator and denominator of the wind equation, and their ratio corresponding to the derivative of square of the Alfvénic Mach number with respect to the poloidal spherical angle $\theta$. From this ratio it is clear we have indeed found a smooth crossing. In comparison with solution $c$, the AP is still located at $\theta = 0.87$ rad or $50^\circ$ from the axis of symmetry, but the MFP has moved outward
Table 4.3: Parameters of solutions

<table>
<thead>
<tr>
<th></th>
<th>$F$</th>
<th>$\Gamma$</th>
<th>$\theta_A$ (deg)</th>
<th>$\psi_A$ (deg)</th>
<th>$\sigma_M$ ($r_g$)</th>
<th>$\sigma_A$</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chapter 3</td>
<td>0.75</td>
<td>5/3</td>
<td>50</td>
<td>55</td>
<td>2.53981</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>This work</td>
<td>0.75</td>
<td>5/3</td>
<td>50</td>
<td>60</td>
<td>2.5</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>Chapter 3</td>
<td>0.75</td>
<td>0.12</td>
<td>-1.66651</td>
<td>2.60390</td>
<td>9.85117</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>This work</td>
<td>0.755500</td>
<td>0.271221</td>
<td>-1.27754</td>
<td>2.25711</td>
<td>12.4036</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

1 The parameters given are for solution $c$ in chapter 3. This is the solution we will compare our new result with. The values for the upper seven parameters ($F$ through $\dot{M}$) of the solution in this work are exact, for the lower five ($x^A$, $q$, $p_A$, $\sigma_A$, $\mu$) they are rounded off. Because singular solutions require high precision, the rounded-off numbers are given with six significant digits.

to $\theta = 0.029$ rad or $1.65^\circ$, while the MSP (only ostensible in solution $c$) has moved inward to $\theta = 0.92$ rad or $52.6^\circ$. This angle corresponds to a spherical radius of $5.96 \times r_g$, which is just within the ISCO.

The values of the velocity, the Lorentz factor, the magnetic and electric field strength, the density and the pressure along a reference field line are given in figure 4.2. The poloidal velocity monotonically increases from 0.067 $c$, through 0.11 $c$ at the MSP, to very close to the speed of light at the MFP. The toroidal velocity starts off positive, turns negative after the AP, and returns to positive before the MFP. The Lorentz factor slowly increases from 1.17 at the MSP to a final value of 12.3.

The poloidal and (negative) toroidal magnetic field strengths first increase in magnitude and then decrease, both peaking around the AP. The electric field has the same behaviour. The density and pressure both drop monotonically as is expected for an expanding jet. Just beyond the MFP the jet overcollimates, causing both density and pressure to increase again.

Figure 4.3 shows the geometry, the energetics, and the Alfvénic Mach number of the jet. The reference field line, plotted in the upper left panel, shows that for most of the expansion the jet is parabolic with the height of the jet approximately the radius to the power of $4/3$. At its highest point beyond the MFP, the jet has overcollimated to only 2 per cent of the maximum width, attained just before the MFP. The main difference between this solution and solution $c$ from chapter 3 is that the height-to-width ratio has increased by a factor of 2.5, which means it has a narrower opening angle comparatively.
4.3 Results

The upper right panel of figure 4.3 shows the partition of energy of the jet. After a small drop, the kinetic energy first increases at the expense of the thermal energy, signifying initial bulk acceleration due to a gas pressure gradient. After the flow has cooled, magnetic acceleration takes over and continues to accelerate the flow, also after overcollimation. Only just before the flow hits the axis does the Lorentz factor decrease again with a corresponding increase in the magnetic field strength.

The lower left panel of figure 4.3 shows the opening half-angle \((\pi/2 - \psi)\) of the flow (also compare with figure 4.5) and the causal connection opening angle. Both angles are measured from the axis of symmetry. The opening half-angle shows that after a relatively cylindrical start, the flow widens, before slowly collimating again. The causal connection opening angle is the equivalent of a sonic Mach cone. The flow cannot influence anything outside of the forward pointing cone with this angle. As the velocity increases, this cone becomes more narrow. This plot is almost identical to that for solution \(c\).

The lower right panel of figure 4.3 shows the Alfvénic Mach number, the velocity in units of the Alfvén speed. After a long initial rise, shortly after overcollimation it decreases again. This decrease relates to the leftmost part of figure 4.1 where the numerator crosses zero again, without a corresponding crossing of the denominator, causing a decrease of the Mach number. Because the Mach number is defined in terms of the Alfvén speed, it is not the overall velocity that is decreasing, as there is no corresponding decrease in the Lorentz factor. It is rather the magnetic field strength that increases due to the overcollimation, leading to a higher Alfvén speed. The plot of the Mach number is very similar to that for solution \(c\), but there is a twenty-fold increase of \(M\). Note that in chapter 3 we plotted the square of the Mach number.

4.3.2 An initial exploration of parameter space

In this section we describe our exploration of parameter space. As a starting point we pick our first solution. We change the value of a single parameter \((\theta_A, \psi_A, \sigma_M, \text{or } M)\) until it is no longer possible to obtain a singular solution by fitting \(x_A^2, q, \text{and } p_A\). The effect of changing these four parameters on the height of the MFP, AP and MSP is given in figure 4.4.

This plot also shows a physical reason why the solutions do not fill up the whole of parameter space. For high values of \(\psi_A\), and for low values of \(\sigma_M\) and \(M\), the MSP approaches the AP. Beyond the location where they coincide, no further solution is possible.

If \(\theta_A\) and \(\psi_A\) are changed together so their sum is roughly constant, the range of solutions spans approximately 45°. If they are changed separately, the range is significantly smaller. \(\sigma_M\) has the largest spread for this case, ranging from 0.4 to
Figure 4.1: The reference solution showing both an MFP and MSP. The red and blue lines show the numerator and denominator of the wind equation respectively, while the black line shows their ratio. The latter determines the total bulk acceleration of $M^2$ with respect to polar angle $\theta$. The vertical line shows the location of the AP. The vertical axis is a ‘scaled logarithm’ of the plotted parameters, i.e., $y = \text{sign}(x) \log_{10}[1 + \text{abs}(x)/10^{-12}]$ to clearly show the variables over many orders of magnitude. At low $\theta$ a small change in angle corresponds to a vast change in height, also contributing to the near vertical change of sign of the bulk acceleration.

above 60, monotonically increasing the height of the MFP. $M$ has the range $4–12 M_\odot$, spanning most of the mass range of astrophysical black holes for an AP distance of $\sigma_A \approx 74$ km.

Figure 4.4 gives the bounds for our solutions. Since we connect the start of the particle acceleration region with the MFP, we are mainly interested in its location within the jet. In the solutions found so far, the height of the MFP ranges from $10^4 – 10^8 r_g$. Because this work is by no means an exhaustive exploration of parameter space, this range provides only a preliminary indication. Nevertheless, it already allows us to model a variety of sources. The best way to change the height of the MFP seems to be adjusting $\theta_A$. The height monotonically decreases while increasing this parameter. Another possibility is decreasing $\sigma_M$. $\psi_A$ and $M$ seem to have little effect over a wide range of values. The MSP should lie in or near the corona, thus locations too far away and too close to the black hole should be avoided. The height of the MSP ranges from $2 – 10 r_g$. This lower value is very close to the Schwarzschild radius and justifies the inclusion of the pseudo-Newtonian potential. The parameter
4.3 Results

Figure 4.2: Various physical quantities that result from integrating the wind equation depicted in figure 4.1, plotted against polar angle $\theta$. The MFP, AP and MSP are marked. Panel (a) shows the Lorentz factor ($\gamma$), the poloidal ($V_p$) and toroidal velocity ($V_\phi$) of our canonical solution. Please note that the toroidal velocity starts out positive, turns negative just beyond the AP (the dotted line) and then becomes positive again later in the outflow. Panel (b) shows the electric field energy ($E^2/8\pi$), the toroidal magnetic field energy ($B_\phi^2/8\pi$), the poloidal magnetic field energy ($B_p^2/8\pi$), the density ($\rho c^2$) and the pressure ($P$). These have all been divided by the scaling factor $B_0^2 \alpha^{\alpha-2}$, where $B_0$ is a reference magnetic field strength and $\alpha$ is the square of the cylindrical distance of the AP in terms of a reference length ($\alpha = \omega^2 A / \omega^2_0$), as defined in VK03. Although the square is plotted, $B_\phi$ is negative everywhere in this model.
Various physical quantities that result from integrating the wind equation depicted in figure 4.1, now plotted against cylindrical radius \( \varpi \) instead of \( \theta \). These plots are similar to those in figure 4 of chapter 3. Panel (a) shows the geometry of the field line. Panel (b) shows the magnetic energy \( S \equiv -\varpi \Omega B_\varphi / \Psi A c^2 \), the thermal energy including the rest mass \( \xi \), and the kinetic energy \( \left( \gamma - 1 \right) \xi \). Panel (c) shows the opening half-angle of the outflow \( \pi/2 - \psi \) and the causal connection opening angle \( \arcsin \left( 1 / \gamma \right) \). Panel (d) shows the cylindrical radius in units of the ‘light cylinder’ radius \( x \) and the Alfvénic Mach number \( M \). The MSP, AP and MFP (from left to right) are indicated for each quantity.

\( M \) has the largest effect on the location of the MSP.

The effect of changing \( M \) on the geometry of the field lines is shown in figure 4.5. This plot shows only the single reference field line of a solution. It is clear the height of the MSP can be smaller than the Schwarzschild radius, but the spherical radius can not. This geometry explains why the height of the MSP can be smaller than \( 2 r_g \) in figure 4.4. As shown by the lower right panel of this figure, the solutions with a higher black hole mass collimate first, and reach a lower total height (in \( r_g \)).

### 4.3.3 Self-similarity

By including gravity in the relativistic equations of MHD, the assumption of self-similarity along the field line is broken. Near the black hole, the flow is non-relativistic.
4.3 Results

Figure 4.4: Examples of how the heights (in gravitational radii) of the solutions' three singular points change as each of the four principal free parameters is changed: red line: MFP; blue line: AP; black line: MSP. The initial solution (see table 4.3) is indicated by the vertical black line. The parameters varied are as follows: panel (a): poloidal spherical angle of the Alfvén point ($\theta_A$); panel (b): the angle the field line makes with the disc at the Alfvén point ($\psi_A$); panel (c): magnetisation ($\sigma_M$); and panel (d): black hole mass ($M$) in units of $M_\odot$. The horizontal axis in each of these plots is linear, not logarithmic. Note the dramatic excursions in the height of the MFP (3 orders of magnitude) with only modest changes in the free parameters. Note also that the MSP and AP heights are still outside the black hole horizon (see figure 4.5); i.e., even though the height of the MSP $< 2r_g$, the MSP spherical radius for these solutions remains outside the black hole horizon ($r > 2r_g$).

and the wind equation reduces to the form of VTST00, which is self-similar. Far away from the centre, the gravitational energy is negligible and the wind equation assumes the form of VK03, which is also self-similar. This combination means that along the integrated field line, the flow transitions from one form of self-similarity to another. It should be noted that the reference field line we integrate will always be continuous.

Since one field line does not constitute a jet, we would like to be able to describe the flow in a region around our reference field line. Changing field lines is done by varying the parameter $\alpha = \sigma_A^2 / \sigma_0^2$, where $\sigma_A$ is the cylindrical radius of the Alfvén point for a specific field line, and $\sigma_0$ is a reference length. By choosing this reference length to be the cylindrical radius of the Alfvén point for our reference field line, without loss of generality that field line has $\alpha = 1$.

Changing $\alpha$ to select different field lines then becomes equal to changing $\sigma_A$ or $M$. Since this parameter is one of our model parameters, the other model param-
Figure 4.5: The effect of changing $M$ on the field line geometry. The black line is our canonical solution in Table 4.3. $M$ ranges from $7 - 10 \, M_\odot$ in steps of $0.5 \, M_\odot$. In panel (a), the MSP and AP are indicated by the black dots. The arrows point to the AP. In panel (b) the MFP is indicated by black dots. Note that the height of the MSP (indicated at the lower end of the line) can be smaller than the Schwarzschild radius, but its spherical radius always remains outside the horizon.

Parameters have to be fit in order to obtain a solution passing through all three singular points. Although the best result would be obtained by changing all parameters simultaneously, for simplicity we limit ourselves to two. We are free to choose any two parameters, and when using $x_A^2$ and $q$ the field lines do not cross, which is required for self-similarity. Because $\theta_A$ and $\psi_A$ are therefore constant for all field lines in a region around our reference field line, at the AP self-similarity is exact. We will define this region, where the field lines do not cross and satisfy self-similarity to within a specified error, as a flux tube.

To show that the non-crossing of field lines holds for a large part of parameter space, we have performed this test at different parameter values. Figure 4.6 shows the deviations from self-similarity of this solution by dividing several field lines around a reference field line by this central field line. For perfect self-similarity this ratio should be a constant. In our case self-similarity is maintained for the majority of the jet. Only very close to the MSP do the deviations become more pronounced, due to gravity. These deviations are also demonstrated by the MSP occurring at slightly different angles for different values of the black hole mass $M$ (see figure 4.5). If we allow a maximum deviation of 10 per cent, which in this case will occur at the MSP.
Figure 4.6: The ratios of the radial size of several field lines with regularly increasing $\sigma_A$ fitted with $x_A^2$ and $q$. The bottom line has $\alpha = 0.64$ and the top line has $\alpha = 1.6$ (where $\alpha \equiv \sigma_A^2 / \sigma_0^2$). The horizontal dashed lines show the values for exact self-similarity. The deviations from self-similarity only get pronounced near the MSP. The shaded region shows the width of the flux tube for which the combined deviation is smaller than 10 per cent beyond the MSP. The parameters of the reference solution are $x_A^2 = 0.01, \sigma_M = 0.01, q = 0.01, \pi_A = 5.08 r_g, \mathcal{M} = 10 M_\odot, \theta_A = 0.952915, \psi_A = 89.1119$. The Lorentz factor at the MFP is just above 11, showing this solution is relativistic.

since we will terminate our solutions there, the width of a flux tube would be about 0.30 of the radius of the central field line, which is a significant fraction.

Beyond the AP, deviations can be caused by changes in the fitting parameters to obtain singular solutions for the different field lines. The bigger the differences in these parameters, the greater the deviations and the narrower the resultant flux tube. The required changes in the parameters depend on the location in parameter space, and so the allowed width depends on the parameter values chosen. For the parameters of figure 4.6, the deviations beyond the AP are very small. While our equations do not strictly satisfy self-similarity, we have shown that it holds very well and that determining the width of the flux tube where it does is straightforward.

### 4.4 Discussion

By including gravity in the MHD equations, we are able to extend earlier solutions that crossed the MFP and AP to also go through the MSP, allowing us to describe a
jet that crosses all three singular points. Because of this lower boundary condition, we now have a reliable description of the jet below the AP. This description gives a smooth solution from very near the central object out to the point of overcollimation, using a single partial differential equation to describe all regimes. This approach produces a workable physical model, allowing us to tie the jet properties to the conditions at the base, and providing a self-consistent determination of the start of the particle acceleration region.

At the same time, however, the addition of gravity also violates the conditions for self-similarity. The reason why a single self-similar relativistic flow equation, with gravity, has not been derived is because relativistic flow has one scaling with radius, while gravity has another. Our $C^\infty$-continuous bridging method has not changed that situation: while describable with a single, continuous equation, our solutions in the relativistic part of the flow without gravity will have different dependencies on the radius parameter $\alpha$ than in the non-relativistic part near the black hole that includes gravity. That is, at least one term in our wind equation will have a dependency on the radius parameter $\alpha$.

In effect our modified wind equation creates two regions with self-similar geometry, but with different self-similar dependencies in each region. Because our focus is on the bulk acceleration and collimation of jets in relativistic sources, our approach to this issue is to choose the self-similarity of the relativistic VK03 equations and restrict the different dependency on $\alpha$ to the gravity term only. That is, our solutions will not be strictly self-similar in the low-speed part of the flow with gravity (the VTST00 regime), but they will be self-similar far from the black hole in the VK03 regime (see figure 4.6). This highlights another advantage of this approach: since the gravity term is an algebraic one only (not involving any derivatives of the flow parameters with respect to either radius or polar angle), all the physical radial and angular dependence of the original equations will be preserved.

The different self-similarities within a single solution is not unique to our approach; it will be true of any method that attempts to bridge self-similar solutions with and without gravity. In this regard, interpretation of the solutions will be an important aspect of this study. There are two main issues to consider.

First, while accretion discs and jets may have a self-similar character, both simulations and observations show that their activity is concentrated in specific regions. For example, much of the accretion, and therefore jet, power is generated near the disc inner edge. And features like the one we seek (a strong collimation shock in the flow) will occur primarily at a specific point in the jet, and be observed as such, rather than being spread over a large range in radius. Therefore, we shall concentrate on only one bundle (‘flux tube’) of magnetic field and stream lines and assume that this flux tube is anchored near the disc inner edge and that it passes through the most
important features of the jet. This will limit the range in \( \alpha \) in which we are interested.

Second, in order that the solutions remain reasonably self-similar-like, we shall choose solutions in which the field lines do not cross within that flux tube. By choosing a specific combination of the free parameters to fit a solution with three singular points, the field lines around our reference field line behave well. We have found that selecting \( \chi_A^2 \) and \( q \) as fitting parameters for the singular points produces results that satisfy self-similarity well within a flux tube of finite width.

With our method, therefore, we will restrict our solutions to a limited, and conservative, region of parameter space. The flow in our solutions must remain reasonably non-relativistic in the VTST00 part of the flow (when gravity is important), and we must consider only flux tubes of narrow enough width that field lines satisfy self-similarity to within a specified error in this region with gravity.

Figure 4.7 can be used to see whether the assumptions of the two regimes are not violated for a particular solution. The kinetic energy \((\gamma - 1)\) should be negligible near the black hole, while the gravitational potential energy \(\left(\frac{\mathcal{G}M}{c^2 r}\right)\) should be significant, satisfying the conditions of the non-relativistic regime and demonstrating the importance of gravity. Around the Alfvén point the kinetic energy overtakes, without becoming relativistic. This is the bridging region where gravity becomes unimportant. Only beyond the Alfvén point does the flow become truly relativistic, while gravity becomes negligible, satisfying the conditions of the relativistic regime.

In this work we have only made a cursory exploration of the full parameter space, and it seems likely that part of that space will allow for solutions with smaller MFP values. The location of the MSP falls within the range of a radiatively-inefficient accretion flow (RIAF) disc model. Also, the velocity at the MSP, around \( c/3 \), is very reasonable. These features make it plausible that we will be able to match the conditions to an accretion flow as well as to the size of the jet base as indicated by fits to the spectrum (Markoff et al. 2005), allowing us to determine the location of the shock region from the conditions at the base of the jet.

It is important to note that since our assumption of non-relativistic flow (when gravity is important) eliminates the part of parameter space that produces relativistic flows close to the black hole, we exclude solutions that may relate to physical sources. Future work would include a more general model that allows relativistic flow in the gravitational field. However, the model in this paper provides a good description of the jet structure near the MSP when the flow is non-relativistic.

We should also note that the gravitational field we chose to include here is, at best, a pseudo-Newtonian (or pseudo-Schwarzschild) one. Our solutions, therefore, do not take into account any metric rotation that would occur near a Kerr black hole, for example. However, our method might be able to hint at the presence of a Kerr hole, if, for example, after fitting to broad-band data, we find that the footpoint of the field
line may be significantly less than the ISCO of a Schwarzschild black hole ($\ll 6 r_g$) or that the required magnetisation is significantly greater than would be typical at the ISCO of a normal Keplerian disc.

Another issue neglected by our computations (and, indeed, by all steady state analyses) is the effect of instabilities on the accelerating jet flow. We will briefly discuss fluid and MHD instabilities in the weak-field limit and then MHD instabilities in the strong-field limit as well. Figure 2b compares the relative strengths of fluid dynamical forces (i.e., pressure $P$) with electromagnetic forces (e.g., $(B^2 + E^2)/8\pi$). We see that the former dominate only below the Alfvén point, in the slow magnetosonic region, so it is only there where we expect Kelvin-Helmholtz (KHI) or magneto-rotational instabilities (MRI) possibly to be important. Indeed, the MSP itself may lie in the atmosphere of a turbulent accretion disk, giving rise to a somewhat more complex structure in the sub-Alfvénic region than that assumed here. Nevertheless, recent two- and three-dimensional simulations of relativistic jets launched from turbulent accretion disks (McKinney 2006; McKinney & Blandford 2009) show that MHD jets similar to those investigated here (including a well-defined classical slow point) do indeed form and are not disrupted by weak-field instabilities near the accretion disk. Above this point, like other outflowing magnetospheres (solar, pulsar, etc.), the dynamical forces in the flow are so dominated by electromagnetic forces (eventually by many orders of magnitude) that any weak-field instabilities will be strongly suppressed, or at least unimportant in affecting the kinematics of the accelerating jet.

However, strong-field instabilities (in particular, the current-driven helical kink [CDI]) need to be looked at more closely. Nakamura & Meier (2004), for example, showed that in non-relativistic jet flow rotation velocities well above the Alfvén speed can suppress the helical kink, as can a steep external pressure gradient. They speculated that relativistic flow would further reduce the development of kinks, but a similar detailed relativistic study has not yet been performed. Again, as one of the few relativistic three-dimensional MHD simulations, McKinney & Blandford (2009) can give some insight into the behaviour of real MHD jets in the strong-field region. While the accelerating jet appears to be affected somewhat by helical kinking in that study, it does not appear to be disrupted by the CDI. It remains a viable jet well beyond the classical fast point (where the rotational speed should be significantly super-Alfvénic. However, no three-dimensional simulation so far has followed the acceleration out to the MFP region, let alone for long model times to achieve a quasi-steady state. And, detailed numerical parameter studies are even further in the future. So, the question of whether MHD jet acceleration can be largely stable to strong-field MHD instabilities is still an open question.
Figure 4.7: Comparison of the gravitational potential energy and the kinetic energy along the jet of the bottom solution detailed in table 4.3. With this plot we can select solutions where the gravitational energy $\frac{GM}{(c^2 r)}$ is important close to the black hole, whereas the kinetic energy $(\gamma - 1)$ only becomes relativistic further along the jet.

4.5 Conclusions

We have shown that it is possible to extend the time independent, semi-analytic solutions of chapter 3 to solutions that include gravity using a $C^\infty$-continuous bridging method, connecting two valid regimes with a smooth transition region. We also extend the MHD wind/jet equation to include a pseudo-Newtonian potential for gravity. For the first time these solutions cross all three singular points and describe a relativistic jet from the vicinity of a central black hole to the point where the flow hits the axis of symmetry. Although these solutions do not satisfy strict self-similarity, the deviations within a flux tube of a certain width can be estimated and controlled, and thus a physically relevant model can be constructed.

We have discovered a solution crossing all three singular points with parameters very close to one from our previous paper. This new solution shows the relative ease with which solutions can be found and allows for a comparison. The main differences are the higher distance of the MFP and a significantly higher velocity, both owing to a higher bulk acceleration at the AP. Otherwise the two solutions look very similar, showing that gravity has little influence beyond the region close to the central object. By allowing the creation of an MSP, including gravity enables us to have a reliable description of the flow below the AP.
We have made a cursory examination of parameter space by changing one parameter of a single solution at a time. This exploration gives a preliminary indication of the extent of the solution space, while at the same time showing the effects it has on the solutions. In particular we have found that the poloidal spherical angle of the AP has the most significant effect on the height of the MFP. On the other hand, the ratio of the AP to the light cylinder radius has the greatest effect on the height of the MSP.

The multi-dimensional solution space is constrained by physical and mathematical conditions. Changing only one parameter at a time, when the light cylinder radius approaches the AP, the AP moves outwards, the temperature is increased, or the magnetisation decreased, the jets become wider and shorter, with the MSP approaching the AP. By increasing the cylindrical radius of the AP, the effects of gravity decrease. Eventually this decrease leads to the regime where an MSP cannot be created anymore, approaching the situation without gravity. Despite all these constraints, the solution space allows a wide range of parameter values to be chosen, translating into a wide range of properties, like the location of the MSP and MFP, the velocity, magnetic energy, density and pressure of the flow.

By matching conditions at the MSP to an accretion flow model, the mathematical parameters considered free in this work will be tied to the conditions at the base of the jet and subsumed into the model, with the wide range of properties in the solution space assuring a good fit. Consequently, the height of the MFP, and the start of the particle acceleration region, will be uniquely determined by the conditions close to the central object, providing a self-consistent connection. After integration into a model that can determine the spectral emission of a jet solution, we will be able to ascertain the conditions that best describe the overall spectrum of a given black hole system. In future work we will explore how well this model succeeds in predicting the correct location of the optically thick-to-thin break observed in the broadband spectra of compact jets, hopefully with new insights about the physics of jet launching conditions.

4.A Derivation of the gravity term

To obtain a relativistic form of gravity, we need the following conversions: $\sigma_*$ in VTST00 is equal to $\sigma_0$ in VK03 and $V_*$ similarly corresponds to $\frac{a^{1/4}F_{\gamma M}c}{\gamma \sqrt{\frac{\sigma_0}{\gamma}}}$.
4.B Equations for the initial parameter values

Substituting these into equation (17) of VTST00 produces:

\[
\kappa = \sqrt{\frac{GM}{\sigma_0 V_*^2}} = \sqrt{\frac{GM \gamma^2 x_A^4}{\sigma_0 x_0^{1/2} F^2 \sigma_M^2 c^2}}
\]

\[
= \frac{GM \mu^2 x_A^4 (1 - M^2 - x_A^2)^2}{c^2 \sigma_M^2 F^2 \sigma_M^2 (1 - M^2 - x^2)^2}.
\] (4.3)

The full gravity term then becomes:

\[- \frac{\kappa^2 \sin^2(\theta)}{G} = - \frac{GM \mu^2 x_A^4 (1 - M^2 - x_A^2)^2 \sin(\theta)}{c^2 F^2 \sigma_M^2 (1 - M^2 - x^2)^2 \sigma_M G}.
\] (4.4)

In this expression \((\sigma_A G) / \sin(\theta)\) is equal to the spherical radius, \(r\). To include a pseudo-Newtonian potential, we replace \(1 / r\) by \(1 / (r - r_S)\), where \(r_S\) is the Schwarzschild radius \((= 2GM/c^2)\) and simplify:

\[- \frac{\kappa^2 \sin^2(\theta)}{G} = - \frac{\mu^2 x_A^4 (1 - M^2 - x_A^2)^2}{c^2 F^2 \sigma_M^2 (1 - M^2 - x^2)^2} \left[ \frac{c^2 \sigma_A G}{GM \sin(\theta)} - 2 \right]^{-1}.
\] (4.5)

4.B Equations for the initial parameter values

The Alfvén regularity condition is given by evaluating the wind equation at the AP:

\[
\frac{F^2 \sigma_M^2 (1 - x_A^2)(\sigma_A + 1)^2 \sin(\theta_A)}{\mu^2 \cos^2(\theta_A + \psi_A)} \left\{ -2 \frac{\Gamma - 1}{\Gamma} \frac{(F - 2) (\xi_A - 1) (1 - x_A^2)}{\xi_A x_A^2} \sin(\theta_A)
\right. \\
\left. + 2 \cos(\psi_A) \sin(\theta_A + \psi_A) \frac{\sigma_A + 1}{\sigma_A} + \frac{\sin(\theta_A)}{x_A^2} [(F - 1)(1 - x_A^2) - 1] \right\}
\]

\[
= \left[ x_A^2 - \frac{\sigma_A (1 - x_A^2)}{\sigma_A} \right]^2 - (F - 1) \sigma_A^2 (1 - x_A^2)
\]

\[
-2 \frac{\Gamma - 1}{\Gamma} \frac{(F - 2) \xi_A - 1}{\xi_A} \left[ x_A^2 - \frac{\sigma_A (1 - x_A^2)}{\sigma_A} \right]^2
\]

\[
- \frac{x_A^2}{1 - x_A^2} \left[ \frac{c^2 \sigma_A}{GM \sin(\theta_A)} - 2 \right]^{-1}.
\] (4.6)

We obtain \(\mu^2\) by evaluating the energy equation at the AP:

\[
\mu^2 = \frac{(\sigma_A + 1)^2}{x_A^2 - \left[ x_A^2 - \frac{\sigma_A (1 - x_A^2)}{\sigma_A} \right]^2} \left[ x_A^2 \xi_A^2 + \frac{F^2 \sigma_M^2 (1 - x_A^2)^2 \sin^2(\theta_A)}{x_A^2 \cos^2(\theta_A + \psi_A)} \right].
\] (4.7)
4 Linking accretion flow and particle acceleration in jets. I. New relativistic magnetohydrodynamical jet solutions including gravity

4.C Gravity in the energy equation

The only terms that include the effects of gravity in the energy equation given by (A3) in VTST00 are the Bernoulli constant and the gravity term. Dividing out a common factor of 2, we have:

\[ \varepsilon + \frac{\kappa^2 \sin(\theta)}{G}. \]  

(4.8)

At the AP \( G = 1 \) and \( \theta = \theta_A \), reducing these terms to:

\[ \varepsilon + \kappa^2 \sin(\theta_A). \]  

(4.9)

If we write out \( \varepsilon \) using equation (19) of VTST00, \( G \equiv \sigma / \sigma_A \), and equation (2.7a) of BP82, we obtain (please note that to keep to the notation of the preceding two papers, here the subscript \( 0 \) means the value at the base of the outflow, not the reference values as used in VK03. Unfortunately, \( r_0 \) as used in BP82, is \( \sigma_0 \) as used in VTST00):

\[ \varepsilon = \frac{\kappa^2}{G_0} = \frac{e}{GM/\sigma_0/\sigma_A} = \frac{\kappa^2}{GM/\sigma_A} e. \]  

(4.10)

Since \( e \), the specific energy, is a constant of motion, we can evaluate it at the AP, just like the energy equation in VK03 (equation (2.2) of BP82 with the gravitational potential expanded):

\[ e = e_A = \frac{V_A^2}{2} + h_A - \frac{GM}{r_A} - \frac{\Omega \sigma_A B_{\phi A}}{\Psi_{A A}}. \]  

(4.11)

Here \( r_A \) is the spherical radius of the AP. Substituting equations (4.11) and (4.10) into (4.9), we obtain:

\[ \kappa^2 \left[ \frac{\sigma_A}{G M} \left( \frac{V_A^2}{2} + h_A - \frac{\Omega \sigma_A B_{\phi A}}{\Psi_{A A}} \right) - \frac{\sigma_A}{r_A} + \sin(\theta_A) \right]. \]  

(4.12)

However, \( \sigma_A / r_A = \sin(\theta) \), so the last two terms cancel. The \( \kappa^2 \) term cancels the \( GM \) factor (equation 4.3), leaving no terms that depend on gravity. Thus there is no dependence on gravity in the energy equation in the non-relativistic regime with gravity at the AP and consequently there is no gravity term in the relativistic energy equation at the AP (equation 4.7).
Linking accretion flow and particle acceleration in jets. II. Self-similar Jet Models with Full Relativistic MHD Inertia

Peter Polko, David L. Meier and Sera Markoff

Abstract

We present a new, semi-analytic formalism to model the acceleration and collimation of relativistic jets in a gravitational potential. The gravitational energy density includes the kinetic, thermal, and electromagnetic inertias. The solutions are close to self-similar throughout the integration, from very close to the black hole to the region where gravity is unimportant. The field lines are tied to the conditions very close to the central object and eventually overcollimate, possibly leading to a collimation shock. This collimation shock could provide the conditions for diffusive shock acceleration, leading to the observed electron populations with a power-law energy distribution in jets.

We provide the derivation, a detailed analysis of a solution, and describe the effects the parameters have on the properties of the solutions, such as the Lorentz factor and location of the collimation shock. We also discuss the deviations from self-similarity.

By comparing the new gravity term with the gravity term obtained from a non-relativistic formalism in a previous work, we show they converge in the non-relativistic limit. This convergence shows the approach taken in that work is valid and allows us to comment on its limitations.
51 Introduction

Jets are important building blocks of our Universe. When the supermassive black hole in the centre of a galaxy is activated by accretion, a resulting jet can significantly alter its surroundings, by heating or displacing the ambient medium, affecting the evolution of the galaxy, or transporting angular momentum, possibly changing the spin of the black hole itself. Since the central black hole is in general too small to resolve (but see, e.g., Junor et al. 1999; Hada et al. 2011; Doeleman et al. 2012), we cannot usually observe the conditions around it directly. But as the jet is formed in this region, by observing the jet at larger distances, we can obtain this valuable information indirectly, if we have a method of linking formation conditions with the jet flow further out.

The signature radiation from jets is synchrotron radiation, which provides the radio, and sometimes dominates the infrared to X-ray bands. In the radio the observed spectrum is often flat or slightly inverted, which, for a compact jet associated with weakly accreting hard state, is generally interpreted as the superposition of several synchrotron components from an underlying electron population with a power-law energy distribution (Blandford & Konigl 1979). If there was a population of electrons closer to the black hole with for example a quasi-thermal energy distribution, this change would result in a break in the spectrum, corresponding to the location where the particles are accelerated to a power-law distribution for the first time. Such a break has indeed been observed in several sources, both in AGN (Ho 1999), and X-ray binaries (XRBs) (Corbel & Fender 2002; Gandhi et al. 2011; Russell et al. 2013).

Spectral modelling of the broadband spectrum consistently predicts the height of this location of particle acceleration to lie at ~ 10–1000 \( r_g \), where \( r_g \) is the gravitational radius, with a dependence on the luminosity of the source (Markoff et al. 2001, 2003, 2005, 2008; Migliari et al. 2007; Gallo et al. 2007; Maitra et al. 2009a). High-energy electrons would quickly lose their energy via synchrotron radiation, inverse Compton and adiabatic losses. Since the observed flat spectrum implies the power-law distribution persists for times longer than this cooling time, it seems the particles are accelerated continuously beyond this location (Jester et al. 2001).

A natural way to accelerate particles into a power-law distribution is via diffuse shock acceleration (Fermi 1949; Bell 1978; Drury 1983; Rieger et al. 2007). A jet can be collimated by forces such as the pressure of the surrounding medium, or the magnetic tension of the field lines. If these forces are strong enough, the jet may eventually contract, or overcollimate, guiding the matter towards the centre. If this happens, a resulting collimation shocks can provide the required energy and environment to accelerate the particles into the observed power-law distribution. Since particles seem to be accelerated continuously, this shock would have to be a stable feature in the jet. This condition can be satisfied beyond the modified fast point
(MFP), a singular point in an magnetohydrodynamic (MHD) flow, since at the MFP the flow upstream is causally disconnected from the flow downstream. A shock beyond the MFP is therefore unable to disrupt the flow causing it. Around the MFP the flow can also start to overcollimate. For these reasons we use the MFP as a proxy for the start of the acceleration region. Although previous studies stated that in a relativistic formalism the MFP could only lie at infinity (Li et al. 1992; Vlahakis & Königl 2003b), we showed in an earlier article it is possible to cross the MFP at a finite height (Polko et al. 2010).

By developing an MHD formalism describing a jet, we would like to explore the possibility that the ideas given above can explain the observed jet features. Such a jet model could equate the location of a possible collimation shock with conditions close to the black hole where the jet is formed. In order to be able to have a reliable model close to the black hole, we need to cross two other singular points, the Alfvén point where the flow towards the axis matches the Alfvén velocity, and the modified slow point (MSP), where it matches the magnetosonic slow velocity. Without gravity the MSP cannot form. Since in a relativistic self-similar MHD framework gravity scales differently than the other physical quantities, for a purely self-similar model gravity needs to be neglected. In a previous article by comparing a relativistic formalism with a non-relativistic one including gravity, we showed it is possible to include gravity into the relativistic equations and give a prescription for the region where self-similarity holds to within a specified tolerance (Polko et al. 2013a).

In this article we derive the gravity term from the general MHD equations. We show that the previously derived gravity term is the part representing the kinetic inertia. We will therefore call it the kinetic gravity term for brevity. With this comparison we demonstrate that the previous approach is valid and we can also better explore its limitations. The new model is valid in a wider range of cases, since it also takes the thermal and electromagnetic inertia into account. We will therefore call the newly obtained term the full gravity term for contrast. The new model also keeps track of the fraction of the total energy which is present in the gravitational field.

The observed Lorentz factors of jets in AGN are \( \lesssim 10 \) (Lister et al. 2009), while for XRBs they are found to be \( \lesssim 2 \) (Mirabel & Rodríguez 1999). Our aim therefore is to find solutions with Lorentz factors up to 10 and an MFP height in the range of \( 10–1000 \, r_g \).

In section 5.2 we describe the derivation of the full gravity term, the changes made to the previous model, and the effects these changes have on the solutions. In section 5.3 we give an overview of the solutions we find and a preliminary parameter study determining the range of properties. In section 5.4 we discuss our findings and present our conclusions.
5 Linking accretion flow and particle acceleration in jets. II. Self-similar Jet Models with Full Relativistic MHD Inertia

5.2 Method

In this section we briefly review the preceding work, give the derivation of the full gravity term including the contributions to the inertia, list the modifications to the equations, and describe the approach to finding solutions.

5.2.1 Background

In a previous work (Polko et al. 2013a) we showed it is possible to include gravity into a single relativistic wind equation by comparison of a relativistic formalism without gravity (Vlahakis & Königl 2003a) with a non-relativistic formalism including gravity (Vlahakis et al. 2000, hereafter VTST00). By making a term to term comparison of the wind equations of the two formalisms, we were able to isolate the term equating to gravity and derive a single wind equation relating the region where gravity is important and the flow has non-relativistic velocities, to the region where gravity can be neglected and the flow is relativistic.

While this approach gives solutions that satisfy self-similarity to some degree, it still has several stringent constraints. In relativistic flow, due to the appearance of a characteristic velocity, namely the velocity of light, and corresponding length scale, the light cylinder radius, gravity cannot be included in a self-similar way (Li et al. 1992). Therefore only when the flow has non-relativistic velocities at the base of the jet where gravity is important, are the equations self-similar with gravity. This condition implies either that the gravitational potential is not very strong because the MSP occurs relatively far away from the black hole, or that the temperature of the flow is low, so the initial thermal acceleration is not enough to accelerate the matter to relativistic velocities. Another limitation is the gravity term itself. Since this term is derived from a non-relativistic formalism, it only includes the kinetic inertia, while the thermal and electromagnetic inertias are ignored. Since even the kinetic part misses a relativistic correction (see section 5.2.3), in relativistic flows the gravity term is only a poor approximation of the full gravitational effects.

For these reasons we aim to derive a gravity term taking into account all inertias, and relativistic effects. The approach we use, is to cast the derivative of the energy equation with respect to $\theta$, the poloidal spherical angle, and the transfield equation in the form:

$$A_2 \frac{dM^2}{d\theta} + B_2 \frac{d\psi}{d\theta} = C_2,$$

(5.1)

where $M$ is the Alfvénic Mach number, $\psi$ is the angle a field line makes with the disc, $A_2$, $B_2$, and $C_2$ are functions not containing these derivatives of $M$ and $\psi$. We will use the subscript 2 to refer to functions from the transfield equation, and subscript 1 to refer to functions from the energy equation.
5.2 Method

5.2.2 The new gravity term

In order to derive a fully relativistic gravity term with all inertias, we start with the fully general relativistic equations. Making the assumptions of time independence, axisymmetry, and self-similarity, the energy equation along each magnetic field line gives a conserved quantity, the Bernoulli constant, or specific total energy:

\[(\mu - 1)c^2 + \mu \Phi = \text{constant},\]  

(5.2)

where \(\mu c^2\) is the total energy-to-mass flux ratio, or specific internal energy of the plasma, including rest mass, \(c\) is the velocity of light, and \(\Phi\) is the gravitational potential (Meier 2012). If gravity is neglected, \(\mu = \text{constant}\), since \(\Phi = 0\), but if gravity is taken into account \(\mu\) becomes a variable. For ease of notation, we define the Bernoulli constant as:

\[\mu' = \mu + \frac{\mu}{c^2} \Phi = \mu(1 + \frac{\Phi}{c^2}),\]  

(5.3)

so far away from the black hole where gravity is unimportant, \(\mu\) approaches this constant. With the gravitational potential given by:

\[\Phi \equiv \frac{r_g}{c^2} = -\frac{GM}{c^2} \sin(\theta) \omega_A G,\]  

(5.4)

where \(r_g\) is the gravitational radius, \(r\) the spherical radius, \(G\) is the gravitational constant, \(M\) is the mass of the black hole, \(\omega_A\) is the cylindrical radius of the Alfvén point, and \(G\) is the cylindrical radius in units of \(\omega_A\). Although in this section the gravitational radius, \(GM/c^2\) is still explicit, in the results below we absorb it into \(\omega_A\) so this distance is expressed in gravitational radii. A subscript \(A\) denotes a value at the Alfvén point. Using equations (5.3) and (5.4) we can calculate \(\mu'\) at the Alfvén point and consequently \(\mu\) at every point along the field line using equation (5.3).

If we substitute (5.3) into the energy equation and derive the resulting equation with respect to \(\theta\), we obtain an additional term to \(C_1\), which we denote by \(C_1^+\) and is given to first order by:

\[C_1^+ = -\frac{GM \sin(\theta)}{c^2} \left[ \frac{\xi^2 \chi_A^4}{F^2 \sigma_M^2} \frac{\cos^2(\psi + \theta)}{\sin^2(\theta)} + \frac{M^4}{G^4} \right],\]  

(5.5)

where \(\xi c^2\) is the specific relativistic enthalpy, \(x\) is the radius in units of the light cylinder radius \((x = \chi_A G)\), \(F\) is the parameter that controls the current distribution, and \(\sigma_M\) is the magnetisation parameter. The gravitational force in the transfield equation is given by:

\[f_G = \left(\gamma \rho_0 + \frac{\xi}{c^2}\right)(\nabla \Phi \cdot \hat{n}),\]  

(5.6)

103
where \( \gamma \) is the Lorentz factor, \( \rho_0 \) is the matter density, \( \hat{n} \) is the unit vector in the transfield direction, and \( \mathcal{E} \) is the energy density, which under the assumption of flat space-time is given by:

\[
\mathcal{E} = \gamma(\gamma - 1)\rho_0 c^2 + \gamma^2 \left\{ \frac{1}{\Gamma - 1} P - P + \frac{1}{8\pi} \left( B^2 + E^2 \right) \right\},
\]

(5.7)

where \( \Gamma \) is the polytropic index, \( P \) is the pressure, \( B \) is the magnetic field, and \( E \) is the electric field (Meier 2012). If we fill in the expressions for these quantities and use the correct scaling for the transfield equation, we obtain the addition to \( C_2 \):

\[
C_2^+ = \left\{ \frac{x^4}{F^2 \sigma_M^2} \left[ \frac{\mu^2 x^2 (1 - M^2 - x^2)}{M^2} \left( 1 - M^2 - x^2 \right)^2 \right] + \frac{\mu^2 x^2 (1 - G^2)^2}{2G^2 \left( 1 - M^2 - x^2 \right)^2} \right\}
\]

\[
+ \frac{1}{2} \left( 1 + x^2 \right) \frac{\sin^2(\psi + \theta)}{\cos^2(\psi + \theta)} \left\{ \frac{\mathcal{G} \cos^2(\psi + \theta)}{c^2 \sigma_M} \right\}. \quad (5.8)
\]

Solving for \( \frac{dM^2}{d\theta} \) the wind equation is given by:

\[
\frac{dM^2}{d\theta} = \frac{C_1 B_2 - C_2 B_1}{A_1 B_2 - A_2 B_1}, \quad (5.9)
\]

so the addition of the gravity term including all inertias, which we will call the full gravity term, \( \mathcal{G}_{\text{full}} \), to the numerator of the wind equation is then given by \( C_1^+ B_2 - C_2^+ B_1 \), where:

\[
B_1 = \frac{M^4}{G^4}, \quad (5.10)
\]

and:

\[
B_2 = \left[ (1 - M^2 - x^2) + M^2 \sin^2(\psi + \theta) \right] \frac{\sin^2(\theta)}{\cos^2(\psi + \theta)}, \quad (5.11)
\]

are the scaled components from the energy and transfield equation, respectively. The gravity term is then given by:

\[
\mathcal{G}_{\text{full}} = -\frac{\mathcal{G} \cos^2(\psi + \theta)}{c^2 \sigma_M} \left\{ \frac{\mu^2 x^4}{F^2 \sigma_M^2} \left( 1 - M^2 - x^2 \right)^2 \right\}
\]

\[
+ \frac{\mu^2 x^4}{F^2 \sigma_M^2} \left( \frac{G^2 - 2M^2 - x^2}{G^2 (1 - M^2 - x^2)^2} \left( 1 - x^2 \right) \right)
\]

\[
+ \frac{\mu^2 x^4}{F^2 \sigma_M^2} \left( \frac{G^2 - 2M^2 - x^2}{G^2 (1 - M^2 - x^2)^2} \right) \cos^2(\psi + \theta)
\]

\[
- \frac{x^4}{F^2 \sigma_M^2} \left( \frac{\Gamma - 1}{\Gamma} \cdot \mathcal{E} \right) \left[ \frac{1}{(1 - M^2 - x^2)^2} + \frac{1}{2} \frac{M^2 (1 - G^2)^2}{(1 - M^2 - x^2)^2} \right] \cos^2(\psi + \theta)
\]

\[
- \frac{x^4}{F^2 \sigma_M^2} \left( \frac{\Gamma - 1}{\Gamma} \cdot \mathcal{E} \right) \left[ \frac{1}{(1 - M^2 - x^2)^2} \right] \cos^2(\psi + \theta)
\]

\[
+ \frac{1}{2} \frac{M^4}{G^4} \left( 1 + x^2 \right) \sin^2(\theta) \right\}. \quad (5.12)
\]
5.2 Method

5.2.3 Comparison with kinetic gravity term

When we compare the kinetic gravity term obtained in Polko et al. (2013a):

\[ \mathcal{R}_{\text{kin}} = \frac{GM \sin(\theta)}{c^2} \mu^2 \sigma_A^2 \left(1 - M^2 - x_A^2\right) \left(1 - M^2 - x^2\right)^2, \]

(5.13)

with the full gravity term given above, we can see the kinetic gravity term corresponds to the first line of equation (5.12), apart from a factor \((1 - x^2)^2\). For small \(x\) this factor has no effect. The first line of equation (5.12) corresponds to the kinetic energy contribution to \(\mathcal{E}\), which is why we will refer to the gravity term in Polko et al. (2013a) as the kinetic gravity term. The other lines correspond to the thermal, electric and magnetic energies. If these energies are unimportant and \(x\) is small (i.e., the field line lies well within the light cylinder radius), the two gravity terms are similar in value. For solutions with \(x\) close to 1, the kinetic gravity term is a poor approximation, as it will always overestimate the strength of gravity.

The scaling with \(\alpha (\alpha \equiv \omega A / \omega A_0\), where \(\omega A_0\) is a reference length) is the same for both terms. Therefore the full gravity term violates the self-similarity assumption as well. For this reason we postulate that the field lines controlling the location of the MFP originate in a small region of the accretion disc and can be approximated by a flux tube. The width of this flux tube then determines the deviation from self-similarity. We therefore argue that our solutions are valid within a flux tube of a certain width, as laid out in Polko et al. (2013a).

5.2.4 Effects of the full gravity term

There are several changes we have to make to the equations in order to be self-consistent when using the full gravity term. Due to the fact that \(\mu\) is no longer a constant along a field line, we have to calculate it at every step using equation (5.3) from \(\mu'\) calculated at the Alfvén point. Since the potential is negative, \(\mu\) decreases outwards, eventually asymptotically approaching \(\mu'\). Going towards the black hole, \(\mu\) can increase without limit. Since the Lorentz factor of the flow is proportional to \(\mu\), it is possible for the Lorentz factor to decrease when the flow moves outwards. This decrease may sound counterintuitive since the flow is accelerating, but this acceleration is in the poloidal plane, while the Lorentz factor also includes the velocity in the azimuthal direction. The decrease of the Lorentz factor is thus interpreted as the initial relativistic velocities in the azimuthal direction gradually changing into a mainly poloidal flow.
5 Linking accretion flow and particle acceleration in jets. II. Self-similar Jet Models with Full Relativistic MHD Inertia

5.2.5 Approach to finding solutions

Since we want to cross both the MFP and MSP, and these crossings correspond to one boundary condition each, we need to fit two parameters to ensure a smooth crossing. We have chosen to use \( x_A^2 \) and \( q \) for these fitting parameters, as this combination ensures field lines do not cross when we populate the jet by changing the free parameter \( \varpi_A \), which acts as a scaling parameter. By including gravity we have introduced a new characteristic length (the gravitational radius), resulting in the parameter \( \sigma_A \), which relates the cylindrical radius of the Alfvén point to this length. While the simple scaling of true self-similarity is therefore no longer possible, we have found that by changing \( \varpi_A \) we can mimic this scaling to a high degree (see section 5.3.3).

When we have set the free parameters, \( \theta_A, \psi_A, \sigma_M, \) and \( \varpi_A \), we change for example \( x_A^2 \) until the MFP has been crossed. Then we change \( q \) until the MSP has been crossed, losing the MFP crossing. In this way we converge to a solution that crosses both the MFP and MSP.

A key issue in this approach is finding an initial solution. Once obtained, it is possible to find additional solutions by making small steps in parameter space, avoiding any regions where solutions do not exist. Because of the diminishing effect of the \( (1 - x^2) \) factor, we found there were no solutions around the parameter values of the solution described in Polko et al. (2013a), which had a high value of \( x_A^2 \approx 0.75 \). Since all equations, including the gravity term, revert to their non-relativistic form in the appropriate limit, we have adjusted the parameters of the solution given in figure 4 of VTST00 and indeed found a solution nearby (see table 5.1). See section 5.3.1 for further details.

We found that in general the required precision to obtain a solution where the numerator and denominator of the wind equation crossed zero within the same integration step exceeds quadruple precision and involved prohibitive convergence times. For these reasons, when the denominator approached zero first, effectively ending the integration, we extrapolated the numerator and denominator and treated as a solution those integrations where the relative difference between the angle where numerator and denominator crossed zero was smaller than \( 10^{-4} \) radians.

5.3 Results

We will discuss the first solution found using the full gravity term. Then we present an initial exploration of parameter space with a view to finding solutions with properties that correspond to observed systems. We also detail the effects the parameters have on the solution.
5.3 Results

\[ F \quad \Gamma \quad \theta_A \quad \psi_A \quad \sigma_M \quad \sigma_A \quad \mathcal{M} \]

<table>
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<tr>
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\[ x_A^2 \quad q \quad p_A \quad \sigma_A \quad \mu' \]

<table>
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<td>-7.94640</td>
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<td>1.31344</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.1: Parameters of the first solution and the reference solution used in the parameter study. \( q \) is related to the adiabat, \( p_A \) is the value of \( dM^2 / d\theta \) at the Alfvén point, and \( \sigma_A \) the magnetisation function evaluated at the Alfvén point. The values for the upper seven parameters (\( F \) through \( \mathcal{M} \)) of the solution in this work are exact, except \( \sigma_M \) for the first solution, for the lower five (\( x_A^2 \) through \( \mu' \)) they are rounded off, except \( x_A^2 \) for the first solution. Because singular solutions require high precision, the rounded-off numbers are given with six significant digits. For this initial solution \( \sigma_A \) was used as a fitting parameter instead of \( x_A^2 \), since we wanted to ensure \( x_A^2 \) was small.

5.3.1 First solution

Due to the weakening of the gravity term by the \((1 - x_A^2)\) factor, we were unable to find solutions with both an MFP and MSP with parameters similar to those used in Polko et al. (2013a). Since our equations, including the full gravity term, reduce to the equations of VTST00, by using the parameters of a solution that crossed all three singular points in VTST00, we would be close to the values that produce a solution using the relativistic equations, even though this solution would be non-relativistic itself. We converted the parameters of the solution given in the caption of figure 4 in VTST00 to the parameters used in our model. Most parameters are identical, we set \( x_A^2 = 0.01 \) to obtain a solution in the non-force-free limit of VTST00, and for \( \sigma_A \) we used the definition of \( \kappa_{VTST} \) evaluated at the Alfvén point (see equations (2.73), (2.82), and (2.83)):

\[
\sigma_A = \frac{\lambda_{VTST}^2}{\kappa_{VTST} x_A^2} \left( \frac{2 x_A^2 \cos(\psi_A)}{p_A \sin(\theta_A) \cos(\theta_A + \psi_A)} + 1 \right)^{-2},
\]

where \( \kappa_{VTST} \) is the mass loss parameter and \( \lambda_{VTST} \) the specific angular momentum as defined in VTST00. Using the remaining \( \sigma_M \) and \( q \) as fitting parameters, we indeed obtained a solution with parameters given in table 5.1.

The solution found crosses all three singular points as can be seen in figure 5.1. The MFP is located at \( \theta = 6.9^\circ \), the Alfvén point at \( \theta = 60^\circ \), and the MSP at \( \theta = 68^\circ \).

Panel (a) of figure 5.2 shows that the field line geometry is mostly parabolic, with \( \log(z) / \log(r) \approx 1.5 \). In comparison with the solution found in PMM13, the slope is
Figure 5.1: First solution found using the full gravity term showing both an MFP and MSP. The red and blue lines show the numerator and denominator of the wind equation respectively, while the black line shows their ratio. The latter determines the total bulk acceleration of \( M^2 \) with respect to polar angle \( \theta \). The vertical line shows the location of the Alfvén point. The vertical axis is a ‘scaled logarithm’ of the plotted parameters, i.e., \( y = \text{sign}(x) \log_{10}[1 + \text{abs}(x)/10^{-12}] \) to clearly show the variables over many orders of magnitude. The integration is continued beyond the MFP and below the MSP to show \( dM^2/d\theta \) is smooth, but since at the singular points the ratio is almost 0/0, we do not consider these regions for further analysis. Please note that since the angle \( \theta \) decreases with increasing height a negative value of \( dM^2/d\theta \) corresponds to acceleration.

slightly higher and the MSP and Alfvén point lie farther away from the black hole, while the MFP lies significantly closer at a height of 4298 \( r_g \).

Panel (b) shows the kinetic, thermal and electromagnetic energy. The thermal energy drops monotonically to very close to 1, corresponding to thermal acceleration, and leaving the matter in the jet cold. The electromagnetic energy also drops, which means the jet is also magnetically accelerated. Despite these modes of acceleration the Lorentz factor starts out very high and initially drops, before eventually increasing again. This behaviour is caused by the gravitational potential. As the matter climbs out of the potential well, it decelerates, despite the conversion of thermal and magnetic energy. See section 5.4 for a further discussion. The Lorentz factor reaches a value of 1.07 at the MFP, showing this solution is indeed non-relativistic. Due to the low value of the magnetisation parameter, most of the energy is provided by the rest mass of the matter, and the result is a kinetically dominated jet.

Panel (c) shows the causal connection opening angle and the opening half-angle.
5.3 Results

The opening half-angle shows that the jet starts out very wide and slowly collimates, until it overcollimates around the MFP. In contrast the solution in PMM13 started out nearly vertical, widened first and only then started to collimate. The causal connection opening angle is the relativistic equivalent of the angle of a Mach cone for supersonic velocities, describing the cone in which material can be affected by a certain location in the jet. Initially this angle is very small due to the high Lorentz factor and then increases as the flow slows down, the opposite behaviour as in PMM13. Since the Lorentz factor also includes azimuthal velocities, this angle is not centred on the axis of symmetry.

Panel (d) shows the flow crosses the light cylinder radius at a cylindrical radius of \(182 \, r_g\), the location where the blue line has a value of 1. Since the rotational velocity is constant, this means the matter is moving along field lines that are bend backwards. In this solution \(x_A = 0.1\), so the Alfvén point is crossed at a cylindrical radius of \(18.2 \, r_g\), where the Alvénic Mach number is 1.

5.3.2 Exploring parameter space

From the first solution we changed fitting parameters to \(x_A^2\) and \(q\) to obtain a solution with round values for the remaining free parameters. The parameter values of this solution are given in table 5.1. From this reference solution we changed each of the four free parameters in turn to determine the effect they have on the height of the MFP. Figure 5.3 shows that especially \(\psi_A\) and \(\theta_A\) are strongly correlated with this height, with \(\varpi_A\) having a smaller effect. These trends correspond roughly to those found in (Polko et al. 2013a). The allowed range in \(\sigma_M\) is very small for these parameter values before no further solutions can be found. The height of the MSP is correlated with the height of the MFP for \(\theta_A\) and \(\psi_A\) and anticorrelated, albeit weakly, for \(\sigma_M\) and \(\sigma_A\). Since we are restricting ourselves to solutions that have both an MSP and MFP, the convergence of the MSP and Alfvén point for small \(\theta_A\) and \(\psi_A\) can be a possible reason it is impossible to find solutions below certain angles.

Figure 5.4 shows the effect the parameters have on the value of the Lorentz factor at the MFP. Although the Lorentz factor can increase beyond the MFP, since we end our integration at the MFP, we have no information on its evolution there. The most pronounced effect comes from \(\sigma_M\), with a significant increase for only a small change in parameter value. Also \(\theta_A\) and \(\psi_A\) have a clear effect on the Lorentz factor. The parameter \(\varpi_A\), on the other hand, has almost no effect on the Lorentz factor. The Lorentz factors for these solutions are rather small, lying in the range 1.218613 – 1.426178. This is due to the small values of the magnetisation parameter.

If we want to find solutions with smaller MFP heights and higher Lorentz factors, it is clear no single parameter suits our needs. When increased the angles \(\theta_A\) and \(\psi_A\) decrease both the MFP height and the Lorentz factor, while \(\varpi_A\) only affects the MFP.
Figure 5.2: Various physical quantities that result from integrating the wind equation depicted in figure 5.1, now plotted against cylindrical radius $\varpi$ instead of $\theta$. These plots are similar to those in figure 4 of PMM10. Panel (a) shows the geometry of the field line. Panel (b) shows the magnetic energy $(S \equiv -\varpi \Omega B_{\theta} / \Psi_{A} c^{2})$, the thermal energy including the rest mass ($\xi$), and the kinetic energy $[(\gamma - 1)\xi]$. Panel (c) shows the opening half-angle of the outflow ($\pi/2 - \psi$) and the causal connection opening angle ($\arcsin[1/\gamma]$). Panel (d) shows the cylindrical radius in units of the ‘light cylinder’ radius ($x$) and the Alfvénic Mach number ($M$). The MSP, Alfvén point and MFP (from left to right) are indicated for each quantity.

height and $\sigma_{M}$ only the Lorentz factor. A compounding problem is that there is only a limited region in parameter space that allows solutions. Therefore we cannot simply increase $\sigma_{M}$ until we have the right Lorentz factor, as is clearly demonstrated by the limited range in figure 5.4. Thus finding the desired solutions is a case of carefully navigating parameter space avoiding the boundaries. However, figures 5.3 and 5.4 do give a good indication of the properties of these solutions. High $\theta_{A}$, $\psi_{A}$, and $\sigma_{M}$ correspond to a jet that is relatively wide and cylindrical at the base and is strongly magnetised. As a caveat we note that these predictions are based on only a small range of parameter values and the observed trends may not continue beyond these ranges.
5.3 Results

Figure 5.3: Heights (in gravitational radii) of the solutions’ three singular points as a function of the four principal free parameters. The red line shows the MFP, the blue line the Alfvén point, and the black line the MSP. The reference solution (see table 5.1) is indicated by the vertical black line. The parameters varied are as follows: panel (a): poloidal spherical angle of the Alfvén point ($\theta_A$); panel (b): the angle the field line makes with the disc at the Alfvén point ($\psi_A$); panel (c): magnetisation ($\sigma_M$); and panel (d): black hole mass ($M$) in units of $M_\odot$. The horizontal axis in each of these plots is linear, not logarithmic. Note that the MSP and Alfvén point heights are still outside the black hole horizon (see figure 5.5); i.e., even though the height of the MSP $z < 2r_g$, the MSP spherical radius for these solutions remains outside the black hole horizon ($r > 2r_g$).

5.3.3 Self-similarity

Including gravity into the relativistic equations inevitably violates the assumption of self-similarity, which means we cannot simply scale a single solution to populate a jet. By calculating many solutions with different Alfvén point radii, while keeping the other free parameters fixed, we can populate a jet without invoking self-similarity. The reason for choosing $x_A^2$ and $q$ as fitting parameters, is that the field lines of the resulting solutions do not cross. In order to assess our solutions, we would like to know to what extent the jet provided by this method is no longer self-similar. Figure 5.5 shows the poloidal projection of several field lines, both close to the black hole and far away. Figure 5.6 is the ratio of radii of these field lines with respect to the poloidal spherical angle. Perfectly self-similar field lines would be horizontal lines in this plot. Since the poloidal angle of the Alfvén point is fixed, the Alfvén point is geometrically self-similar. The angle of the MFP is almost equal for all these
solutions. The main deviation comes from the MSP. Since the effects of gravity are the strongest there, this is expected. The reason the solutions do not seem to reach the MSP is that in order to calculate a ratio, both solutions have to exist. This plot is thus limited between the highest \( \theta \) of the MFPs and the lowest \( \theta \) at the MSPs. The inability to calculate the ratios is also the reason why the line denoting the MSP is only approximate. Since the actual ratios are unknown, this line is only valid if the ratios satisfy self-similarity at the MSP. While the field lines start to deviate at the MSP, this deviation is within 10 percent, so if we take that as an acceptable limit, all these field lines are allowed within the flux tube.

Apart from geometry, the physical quantities such as the magnetic field strength, density, and temperature also change from field line to field line, due to the changes in the fitting parameters. While in general \( x_\lambda^2 \) only changes by a few percent, the changes in \( q \) can be a factor of several. Since this change can significantly alter the temperature of the matter, limiting the deviations of these quantities from self-similarity can be an additional constraint on the size of the flux tube we consider our solutions in.

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**Figure 5.4:** Lorentz factor at the MFP as a function of the four principal free parameters. The reference solution (see table 5.1) is indicated by the vertical black line. Parameters as in figure 5.3. The vertical axis in each of these plots has the same range for comparison.
5.4 Discussion and conclusions

We have derived an expression for gravity that includes the kinetic, thermal, and electromagnetic inertia, named the full gravity term, from the general MHD equations in the special relativistic limit. This approach is independent of the previous work to include gravity in the relativistic equations, named the kinetic gravity term, but in the non-relativistic limit they converge. This convergence shows that the approach of the bridging solution introduced in Polko et al. (2013a) is valid. This work builds on that approach by including the thermal and electromagnetic inertia, and taking the gravitational potential into account.

The full gravity term has the same $\alpha$-dependence as the kinetic gravity term. Therefore the region where gravity is unimportant and the region close to the black hole have a different self-similar scaling. For this reason we limit our solutions to a flux tube of a certain width, corresponding to the specified deviation of the field lines from self-similarity.

The additional scaling of $(1 - x^2)$ is due to the electric force in the transfield
The ratios of the radial size of the field lines shown in Figure 5.5 with regularly increasing $x_A$ fitted with $\alpha$ and $q$. The horizontal dashed lines show the values for exact self-similarity. The bottom line has $\alpha = 0.34$ and the top line has $\alpha = 2.25$ (where $\alpha \equiv x_A^2/x_0^2$), with respect to the reference field line, denoted by the horizontal black line. The parameters of the reference solution are $x_A = 0.0902$, $\sigma_M = 0.02$, $q = 0.106$, $x_A = 12 r_g$, $M = 10 M_\odot$, $\theta_A = 60^\circ$, $\psi_A = 45^\circ$. The approximate locations of the MFP, Alfvén point, and MSP are marked by the vertical lines.

**Figure 5.6:** The ratios of the radial size of the field lines shown in figure 5.5 with regularly increasing $x_A$ fitted with $\alpha$ and $q$. The horizontal dashed lines show the values for exact self-similarity. The bottom line has $\alpha = 0.34$ and the top line has $\alpha = 2.25$ (where $\alpha \equiv x_A^2/x_0^2$), with respect to the reference field line, denoted by the horizontal black line. The parameters of the reference solution are $x_A = 0.0902$, $\sigma_M = 0.02$, $q = 0.106$, $x_A = 12 r_g$, $M = 10 M_\odot$, $\theta_A = 60^\circ$, $\psi_A = 45^\circ$. The approximate locations of the MFP, Alfvén point, and MSP are marked by the vertical lines.

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5 Linking accretion flow and particle acceleration in jets. II. Self-similar Jet Models with Full Relativistic MHD Inertia

The method we employ to find solutions, relies on the $x_A^2,q$-plane being regular, meaning it is possible to discern whether we need to increase or decrease a param-
eter in order to converge to an MSP or MFP. For certain regions in parameter space the precision we use, is not high enough to avoid numerical artefacts, leading to a checkered plane implying there are many singular points. Since there is not a unique solution anymore, it is impossible to converge to the correct values.

While $\mu$, the sum of the kinetic, thermal, and electromagnetic energy, is a constant if gravity is neglected, in our case it is a variable due to the additional gravitational energy. The gravitational energy increases with height, and because the sum is constant, $\mu$ has to decrease. Since the Lorentz factor is proportional to $\mu$, the counterintuitive result is that the Lorentz factor can decrease with height, even though the flow is accelerating (see figure 5.2). The resolution lies in the fact that the Lorentz factor also includes the azimuthal velocity. Close to the black hole the matter is orbiting at relativistic velocities. When the matter is moving along the field lines, losing energy as it climbs out of the potential well, the azimuthal velocity is slowly converted into the poloidal velocity, causing poloidal acceleration. Only when the gravitational potential has been mostly overcome, does the Lorentz factor increase again due to magnetic acceleration, with a corresponding decrease in Poynting flux.

We performed a parameter study and showed that the results are broadly similar to those in the previous work. The aim is to find solutions with a Lorentz factor up to 10 and an MFP height in the range of $10^5$–$10^6$ $r_g$. While we did not manage to find solutions yet with a Lorentz factor above 1.6 due to the low magnetisation parameter, we have found those with a MFP height of just below $900 r_g$. A more systematic search of parameter space may yield solutions more in line with our aims.

By combining solutions with different radii, we can construct flux tubes that satisfy self-similarity to a high degree. The size of these flux tubes is limited by geometrical deviations and deviations in the physical quantities along different field lines. Despite this constraint we believe it is possible to use these flux tubes to relate the height of the MFP, and therefore the start of the acceleration region, to conditions very close to the black hole. This approach allows for a self-consistent determination of this location and thus of the conditions at the base of the jet from observed broadband spectra of XRBs and AGN.
Discussion and conclusions

In the work presented here we set out to develop a relativistic MHD jet model that includes gravity and, within this model, determine the location of particle acceleration. Here we first provide the general context of the research. Subsequently we give a summary of our main findings and discuss how this research can be used in the future and the possible implications.

There are several important outstanding problems in astrophysics to which our research is relevant. The hot gas surrounding galaxies should cool relatively quickly by emitting X-rays. This cool gas is then expected to fall towards the centre of the galaxy and form stars. However, this star formation is not observed and the gas remains hot despite the energy emitted. The most likely culprit is the jet produced by the central supermassive black hole. By supplying the surrounding matter with copious amounts of kinetic and electromagnetic energy, the jet can prevent the gas to cool down and collapse to form stars.

A second topic of intense study is the origin of ultra-high energy cosmic rays. There are precisely few sites that are energetic enough to provide these particles with the energies observed. Again the most likely source is jets in AGN, with Centaurus A receiving a great deal of attention as a possible source due to its proximity. These cosmic rays are believed to be accelerated by shocks in the jet. In both problems it is very important to know the energetics of the jet and the processes by which particles are accelerated. These are two aspects our research can contribute to.

Another question is the location where this particle acceleration takes place. VLBI observations (Junor et al. 1999; Walker et al. 2008) and spectral fitting (Markoff et al. 2001, 2005; Maitra et al. 2009a) suggest that this location is offset from the black hole at a distance of 10–1000 gravitational radii. We have shown it is possible to connect this location with the conditions close to the black hole.
There are several models for the accretion flow powering the jet. These accretion models provide the boundary conditions for the formation of a jet. Since, based on these boundary conditions, our work produces a unique jet flow solution up until the location of particle acceleration, with the constraints provided by the broadband spectrum, it may be possible to rule out certain accretion flow models.

In the mathematical description of a magnetohydrodynamic flow, such as a jet, three singular points appear. This singular points have important physical implications. The modified slow point, occurring very close to the black hole, provides a convenient location to connect our jet flow solution with an accretion model. This connection provides the necessary information to completely specify the resulting jet. The modified fast point is the location where the flow becomes causally disconnected. Anything happening beyond this location cannot affect the flow closer to the black hole. It is therefore necessary to cross this point before a stable shock, able to accelerate particles, can exist.

We have shown for the first time that it is possible to cross the modified fast point using a relativistic magnetohydrodynamical formalism. Every solution found eventually overcollimates and returns to the axis of symmetry. While this result may be due to the requirement that the modified fast point lies at a finite height, it does allow the possibility of a stable collimation shock capable of accelerating particles. Such a collimation shock is thus an integral element of our formalism. Without gravity, however, it is impossible to have a reliable jet solution close to the black hole and therefore to connect the location of this stable shock to the conditions at the base of the jet. By including gravity into our formalism, we were able to finally make this connection.

By the inclusion of gravity we have broken one of the assumptions of the formalism, the assumption of self-similarity, which means that by a simple scaling any field line can be scaled to every other field line in the jet. By breaking this assumption, a single solution does no longer suffice to specify the entire jet. Instead we need to calculate several solutions, all at different radii, to be able to describe a jet. We have found that choosing two specific model parameters to fit for the modified slow and fast point, namely the ratio of the radius of the Alfvén point and that of the light cylinder radius, and the adiabat describing the temperature of the jet, it is possible to very closely approximate self-similarity. However, the value of the adiabat can vary significantly across the field lines, meaning the matter transported along the outer field lines is colder than it should be compared to the matter on the inner field lines.

While our aim is to find solutions where a shock may form at 10–1000 gravitational radii, so far we have no found solutions much below 900 gravitational radii. Looking at the effects the model parameters have on this height, it seems the best approach to finding solutions with smaller heights, is by moving the Alfvén point.
closer to the disc and having a more cylindrical jet. We have not found solutions with Lorentz factors higher than 1.6, too low for some X-ray binaries and most active galactic nuclei. These low Lorentz factors are due to a low value magnetisation parameter. Since there is little magnetic energy in the jet, the matter cannot be magnetically accelerated to high velocities. Finding solutions with a higher magnetisation parameter is therefore the best strategy to obtain higher Lorentz factors.

This change in distribution has a significant effect on the synchrotron contribution to the overall spectrum. A quasi-thermal distribution has a corresponding exponentially decreasing flux at the high-energy end, strongly localising the frequency of the emission based on the magnetic field strength. In contrast, the accelerated distribution also causes a power-law component above the frequency for which the jet becomes optically thin, with a spectral slope $S_\nu \propto \nu^{-(p-1)/2}$, where $p$ is the power-law index of the energy distribution ($dn = N_0E^{-p}dE$). This power-law component can extend to appreciably high frequencies, with the highest contribution from the location where the particles first get accelerated, due to the higher field strength there. Thus the height of this location determines the contribution to the, in the case of XRBs, X-ray emission from synchrotron emission produced in the jet. For the same observed X-ray flux, a higher height would increase the need for an alternative source of emission, such as inverse Compton. Constraining the height of this location by self-consistently deriving the height of the MFP can therefore give us essential information about which emission mechanisms are important and the prevailing conditions in the jet and disc.

The analytical power of this model truly comes into its own when used as part of a spectral fitting code. This code, although expanded and refined over the years (Falcke & Biermann 1995; Falcke 1996; Falcke & Markoff 2000; Markoff et al. 2001, 2003, 2005, 2008; Maitra et al. 2009a), was originally developed to test the hypothesis that the hot magnetised corona could be the base of the jet, called the jet-disc symbiosis. It determines the spectral contributions from an accretion disc and the jet and takes different radiative processes into account. By comparing the calculated spectrum to those of observed black hole systems, it is possible to constrain several properties of the system. For example, the height of the start of the particle acceleration region regulates the contribution of the jet to the X-ray bands. Since we are able to determine this height self-consistently, we will be able to determine the relative importance of the radiative processes contributing to the X-ray.

However, the jet in this code is described by a hydrodynamical model (Falcke & Biermann 1995; Falcke & Markoff 2000), so the magnetic field does not accelerate or collimate the jet. Moving to a magnetohydrodynamical model has several clear advantages. First, the magnetic fields are treated in a self-consistent way, providing acceleration and determining the geometry of the jet. Second, the height of the
acceleration region becomes a derived quantity and is removed as a free parameter. The additional four free parameters from the MHD model can be tied to the magnetic field strength, the temperature, the density and the initial velocity at the base of the jet, and thus do not contribute to the total number of free parameters. Third, we will be able to calculate the power of the jet independently, giving us a better handle on AGN feedback and cosmic ray energies. Fourth, there are fewer limitations to the environments that can be modelled, as either the velocity, the magnetic field, the gravitational potential, or all can attain relativistic energies. We can therefore apply the code to an even greater variety of sources of both mass scales. In this way we can test the similarities and theoretical scalings among them, extending the fundamental plane of black hole accretion. This should give us new insights into the physics of jet launching.
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6 Discussion and conclusions

Hoewel zwarte gaten zelf zeer tot de verbeelding spreken, hebben ze ook erg interessante gevolgen voor hun omgeving. In dit proefschrift bespreken we één van deze gevolgen, de relativistische straalstromen (jets) die vlakbij het zwarte gat ontstaan en op hun beurt grote veranderingen in hun omgeving kunnen veroorzaken.

Zwarte gaten vallen breed gesproken in twee categorieën. De stellaire zwarte gaten, met een massa van ongeveer 3 – 100 zonsmassa’s, komen verspreid door de melkwegstelsels voor. Deze zwarte gaten zijn voornamelijk de overblijfselen van geïmploedeerde zware sterren. De superzware zwarte gaten, met een massa van $10^5 – 10^{10}$ zonsmassa’s, liggen in de centra van melkwegstelsels. Deze zwarte gaten zijn waarschijnlijk gevormd door de fusie van lichtere zwarte gaten tijdens de versmelting van hun omliggende melkwegstelsels. Er is enig bewijs voor het bestaan van zwarte gaten met een massa tussen deze twee extreem, voornamelijk verwacht in het centrum van bolhopen, maar deze lijken veel zeldzamer te zijn dan hun lichtere en zwaardere soortgenoten.

Straalstromen kunnen verschillende effecten hebben. Rond vormende sterren kunnen ze impulsmoment naar buiten transporteren, waardoor sterren zwaarder kunnen worden dan verwacht op basis van behoud van impulsmoment. De zon heeft bijvoorbeeld maar 0.3% van het impulsmoment van het zonnestelsel, terwijl zij 99.9% van de massa bevat. Als de straalstroom relativistische snelheden bereikt, kan het uitgezonden licht in slechts een nauwe bundel het heelal in worden gestuurd. Door deze concentratie van het licht, bijvoorbeeld in gammastralers, zijn de bronnen over veel grotere afstanden zichtbaar. Een derde effect komt voor bij de straalstromen rond de centrale zwarte gaten in melkwegstelsels. Door de enorme hoeveelheid energie die in deze straalstromen voorkomt, kan het gas rond de kern zo verwarmed raken, of zelfs weggeblazen worden, dat de stervorming in het melkwegstelsel drastisch verminderd.
Om deze effecten te kunnen beschrijven, is een goed begrip van deze straalstromen noodzakelijk.

Helaas zijn de exacte fysische processen waardoor deze straalstromen ontstaan nog niet goed bekend. Er is wel een algemeen geaccepteerd beeld ontstaan van wat er zich waarschijnlijk rond een actief zwarte gat afspeelt. Wanneer er materie richting het zwarte gat valt, verdwijnt dit door behoud van impulsmoment niet direct in het zwarte gat, maar vormt het een snel draaiende schijf eromheen. Omdat de draaisnelheid toeneemt naarmate de materie zich dichterbij het zwarte gat bevindt, wordt de schijf door wrijving op en begint te stralen. De potentiële zwaartekrachtsenergie van de materie wordt dus gedeeltelijk omgezet in kinetische energie en gedeeltelijk in warmte en straling. In de schijf wordt het aanwezige magneetveld versterkt en ontstaan er veldlijnen loodrecht op de schijf. Door de draaiing van de schijf winden de veldlijnen zich op tot een spiraalstructuur en langs deze veldlijnen kan de materie in de schijf ontsnappen.

Door de hoge temperaturen bevindt de materie zich in de toestand van een plasma, een soep van geladen deeltjes. Dit plasma roteert snel rond de veldlijnen en aangezien versnelde geladen deeltjes stralen, wordt ook in deze straalstroom licht uitgezonden. De frequentieverdeling van dit licht hangt af van de energieverdeling van de deeltjes in het plasma. Door waarnemingen aan deze systemen lijkt het erop dat de deeltjes hoger in de straalstroom een hogere energie hebben dan de deeltjes dichterbij het zwarte gat. Dit verschil duidt erop dat de deeltjes in de straalstroom op een of andere manier een hogere energie krijgen, of met andere woorden, worden versneld.

Het mechanisme voor die versnelling dat wij in dit proefschrift bespreken, is dat de deeltjes versneld raken door middel van een schok in het plasma. De deeltjes die door de schok heen bewegen, kaatsen af op magneetvelden die naar hen toe bewegen, waardoor ze versneld raken, onafhankelijk van welke richting ze door de schok heen bewegen. Bij elke kruising van de schok is er een kans dat de deeltjes ontspanten en de versnelling dus ophoudt. De verdeling van de deeltjesenergieën die op deze manier ontstaat, komt goed overeen met de waargenomen verdeling.

Maar waar kan zo’n schok ontstaan? Aangezien de sterkst versnelde deeltjes sneller hun energie kwijtraken dan langzamere deeltjes, terwijl uit waarnemingen de verdeling constant lijkt te blijven, moet de versnelling langsere tijden en over grote afstanden blijven bestaan. Een schok beweegt alle richtingen op en kan dus potentieel de stroom die de schok veroorzaakt, verstoren. Als de stroom op de locatie van de schok echter sneller beweegt dan het snelst mogelijk signaal, kan de schok nooit zijn eigen formatie verstoren. Een manier om een schok te vormen, is door materie op elkaar te laten botsen. Als de diameter van een straalstroom afneemt, zal de materie uiteindelijk weer bij elkaar komen en dat kan een schok veroorzaken. Aan beide voorwaarden, een straalsnelheid hoger dan het snelste signaal en een afnemende
Het hoofddoel van dit proefschrift is het afleiden van een model dat de locatie van deze schok verbindt aan de omstandigheden vlakbij het zwarte gat. Op deze manier is de locatie niet langer een vrije parameter en kan deze informatie geven over eigenschappen als de temperatuur, dichtheid, magnetische veldsterkte en deeltjessnelheid aan het begin van de strangestroom. Om een breed scala aan bronnen en de omstandigheden vlakbij het zwarte gat te kunnen beschrijven, hebben we een nieuw model afgeleid dat zowel relativistisch is als de zwaartekracht beschrijft. Door oplossingen te vinden die op een continue manier door alle drie de singuliere punten die in een magnetohydrodynamische stroom kunnen voorkomen, heen gaan, zorgen we ervoor dat we een betrouwbare beschrijving hebben van het begin van de strangestroom dichtbij het zwarte gat tot de locatie van de schok.

In hoofdstuk 2 geven we een kort overzicht van de geschiedenis van dit type model en beschrijven we de afleiding van ons nieuwe model. In hoofdstuk 3 laten we zien dat het mogelijk is om door het gemodificeerde snelle punt heen te gaan en geven we een gedetailleerde beschrijving van verschillende oplossingen. In hoofdstuk 4 bespreken we de gevolgen van het toevoegen van een kinetische zwaartekrachtsterm, presenteren we oplossingen die door alle drie de singuliere punten gaan en laten we zien hoe de hoogte van schok afhangt van de verschillende parameters. In hoofdstuk 5 beschrijven we de afleiding van een volledig relativistische zwaartekrachtsterm die niet alleen de kinetische, maar ook de thermische en elektromagnetische effecten op de zwaartekracht meeneemt. We voegen ook de gevolgen van de zwaartekrachtspotentiaal toe. Opnieuw vinden we oplossingen die door alle drie de singuliere punten heen gaan en we vergelijken deze met de eerder gevonden oplossingen.

Als we dit model combineren met een programma dat het spectrum van een systeem met een actief zwart gat kan berekenen, wordt het mogelijk uit het spectrum de eigenschappen rond het zwarte gat te bepalen. Door deze eigenschappen te vergelijken tussen sterren en superzware zwarte gaten, kunnen we de verschillen en overeenkomsten tussen deze twee zeer verschillende massaschalen vinden en meer te weten komen over de vorming en gevolgen van deze strangestromen.
A lot has changed in the past few years. Countless people have entered my life and a slightly smaller number has left. Here I would like to pay tribute to those that have made it a memorable time.

First of all, Sera, thank you for giving me the opportunity to do a Ph.D. Without that chance, I honestly don’t know what I would be doing now, but I’m pretty sure it would be worse. Dave, thank you so much for all the discussions about science and everything else in life. I hope one day to give that experience to someone else. Samia, I’m very glad you choose to do your PhD here. It’s been great having someone to share the highs and the lows with. Salomé, thank you for your kindness and your unwavering optimism. Rich, you left far too early, there is still a lot I could have learned from you. Dipankar, you were also gone before I had properly started, but maybe we’ll meet again somewhere in the US.

Theo, you keep surprising me, and always in a good way. Caroline, thanks for the corridor conversations, but I really think it’s about time you finished that painting. Maithili, it was great sharing an obsession with you. I hope we manage a few more in the future. Gijs, for someone with those T-shirts you have a surprisingly cheery disposition. Let me rephrase that: for someone with those T-shirts you have a surprisingly cheery disposition. Martin, well, what to say about you. Based on all the stories I heard beforehand, it was surprisingly fun being your officemate for one and a half years. Although I don’t really think we reached the ‘core’ that often, you certainly have a unique way of leading/disrupting the conversation. Daniela, our resident Pollyannaish polymath. I’m pretty sure in a few years I’ll be able to say “I knew her before she was world famous.” Stefano, feel free to call me for dinner anytime. And the music is great. Oscar, I’m glad you decided to stay at our institute. Dario, your attempts to make Giulia speak English are inspirational. Yi-Jung, I enjoyed the
many long conversations we had. Thijs, thank you for the late night conversations and introducing me to my lovely paranymph. Thijs, although I was quite chuffed I finished my stretch in one go, you made it clear I’m still a rank amateur. Christian, it was great discussing all things typographical with you. Olga, your drawing skills are amazing and your Felix Baumgartner was truly inspired. Danai, you’re an instant +5 mood modifier. I really enjoyed the after-hours conversations. Yvette, thanks for being a good sport. I guess I overdid it a bit. Rik, you rock, don’t ever change. Joel, thanks for introducing me to the Ring des Nibelungen in a very indirect way. If it’s still running, I’m going to have a look. Antonia, I loved sharing my obsession for Multi-Element Radio-Linked Interferometer Network (you know what I mean). I hope we’ll escape Colditz soon. John, overall I really enjoyed the time spent at your house (I still think you have some unfinished business with my landlady). Anna, I still remember that marathon session at your place fondly. Phil, I think back fondly on all our discussions about language and hope we can add a few more. Matt, it was bad timing, but I have some free time coming up. Lucy, wasn’t that Richard III bloke story already news months ago? Adam, I think you’re the only person able to pull off wearing those trousers. Natalie, it’s great having you at the institute, even though you’re occasionally moonlighting for the other ones. To all the Brits above, just make sure next year there will be an institute-wide pancake day. Anton is worth it. Milena, thanks for taking no heed of social conventions and saying wildly inappropriate things. I think you by yourself are responsible for about half of all the stories in our institute. Minou, thank you for everything you’ve done for me. Eva and Nicole, it’s been a pleasure. Ralph, thank you very much for your patience and saving the day when I completely misunderstood or misinterpreted something. I owe you a great debt. Carsten, thanks for all the kindness. Alex, ik ben erg blij dat je een onderdeel van dit instituut bent. Godelieve, dank je wel voor het eindeloze nakijken en voor je hulp bij het organiseren van de NAC. Michiel, bedankt voor de hartelijke gesprekken. Rudy, ik ben blij dat we dat gesprek toch nog gehad hebben. Lidewijde, hartelijk dank voor de hulp bij alle financiële zaken.

Of course our institute is not an island, entire of itself. Via GRAPPA there are many links with the Institute of Physics and NIKHEF. I’ve been lucky enough to meet several of them. Ariana, I know few people more cordial than you. Bogdan, thank you very much for all the discussions. Hamish, your sense of humour takes some getting used to, but I really enjoy your company. Miłosz, thanks for introducing me to Hungarian cinema. György, your skill with cards is uncanny. Balázs, your skill with balls is uncanny. Stephan, Erwin, Claudio, Tino, Giorgos, Koen, Jeroen, Robert, and Arjen have allowed me to indulge in one of my addictions on a regular basis for which I’m deeply grateful. With many thanks to Tri for introducing me in the first place. It was the best possible parting gift.
And now we move to people who may never even read this. I was very lucky to spend four months at the magical place that is Caltech. We may think we do some pretty outrageous things at our institute, but that is child’s play compared to what happens over there. Cathy, no amount of thanks can convey the gratitude I feel for introducing me to S.P.E.C.T.R.E. (it even won me the dry T-shirt contest) and Blacker, and through that, the about hundred other people I met while there. Without you it wouldn’t have been nearly such a social, explosive and insane experience. Alex, thanks for making me feel welcome in your kitchen/living room. Michael, those hourlong sessions of Mafia (that was just one round) will stay with me forever. Oana, thank you for forgetting your bus fare and being a perfect friend. It was really nice exploring Caltech together. Alessondra, because of you I know what the fourth of July means in the US and am also spoilt for life when it comes to fireworks. Laura, your tireless attempts to teach me how to swing a sabre has left me with a lifelong passion. David, I really enjoyed our scientific lunch breaks. Varoujan, every week I looked forward to the breaks and the many outings in and out of JPL you organised. The mars rovers one week, Mount Wilson the next, your phenomenal knowledge of the history and goings-on at JPL were absolutely enthralling. Vena, it was great sharing our love of all things Star Trek and other sci-fi. Aliza, thank you for recognising me on my second first day and immediately dragging me into the library life and the film nights again. Jenn, I was very lucky to have you as a friend, as well as a guide for the tunnel tour. Feynman’s mattress is a legend. Christian, thank you so much for letting me star (well, two lines, so let’s make that co-star) in your Batman film. I really gave student #2 all I got. Joanna, letting me play a rabbi conducting a gay beach wedding just before a tsunami, with as the only instruction “Just do something!” may not have been your wisest decision, but it was definitely one of the highlights of my time there. Kurt, don’t worry, I’ll never tell anyone who exploded that giant hydrogen balloon over Caltech. I’m sure the tesla coil and vortex cannon are still operational and in frequent use. Ilya, watching the Perseids while lying on the rooftop was simply awesome. Samson, thanks for being there when things quietened down and letting me join the laser tag. To everyone else I met, it was great, just great. Silvana, you should know it’s dangerous taking a stranger who doesn’t speak your language on a whirlwind trip through Buenos Aires. I’m glad you did, though. Manuela and Ana Maria, thank you for continuing that apparently South American tradition, and Marie, Huw, Tobi and Padi for finishing the week that ranks as one of the best I’ve ever had.

As mentioned above, after I returned from Caltech for the first time, I delved into a small subculture of people poking each other and liking this. I never realised how addictive this could be, and now I spend a fair amount of time hooked on an electronic machine. Paul was de beste maître de we ons konden wensen. Dat ik
me daar gelijk thuisvoelde, had alles met Alex te maken. Nadat ik zijn gevoel voor humor door had, was het heerlijk tegenover Lorenz op de loper te staan. Ik denk niet dat die sneeuwpop met Henriëtte er ooit nog van komt, dus misschien dat we het maar weer bij warme chocolademelk moeten houden. Helaas is er een tijdperk voorbij, maar het was wel een geweldige tijd. Tarek, it’s rare to meet someone whose company is always an utter joy. Thanks so much for saying “Yes” a lot and showing me a side of Amsterdam I never would have discovered myself. I’m sure we’ll beat that 105 some day. Anna, I’m sorry, I’m so sorry. Jeanice, all the best and good luck. Katja, het was erg leuk wat gedeelde geschiedenis op te bouwen. Ik hoop dat er nog flink wat bij komt. Sacha, Lucian, Niek, Lennaert, Diego, Gati, Eva, Mariska, Eoïn, David, Melanie, Robin, Boy, Daniel, Esther, Laura, Ginger, Susan, Valeria, jullie maken het een feest elke week te kunnen schermen. De geest is uit de fles. Succes dat te beteugelen, Daniël.

And all these are just a tiny fraction of all the people I got to know during my Ph.D. To everyone I forgot, I promise it’s only temporary.

En dan zijn er natuurlijk zĳ die ik al bijna mijn hele leven ken. Aan mijn ouders, dank voor het eindeloze vertrouwen en de steun, vooral op de momenten dat het misschien niet helemaal gerechtvaardigd was. Annette voor het vele contact en voor de constante herinnering dat er ook een wereld buiten de academische is. Zonder jou had ik veel minder ondernomen. Verder Wilko, Cilia, Anneke, Henk, Marian, Henk, Jolanda, Tharra, Mark en María voor het vormen van een geweldig warme familie.

Zo, het is af.